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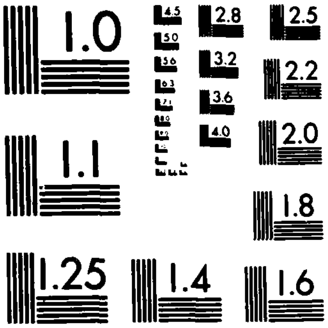

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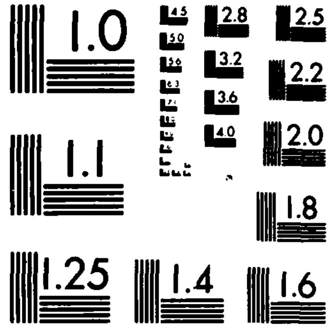

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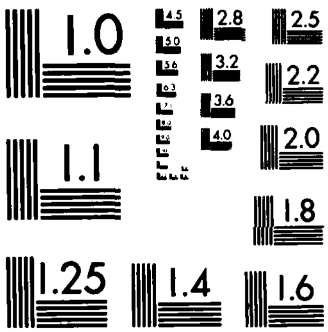

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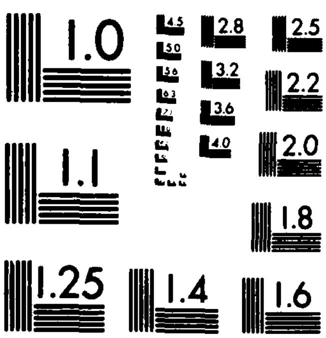

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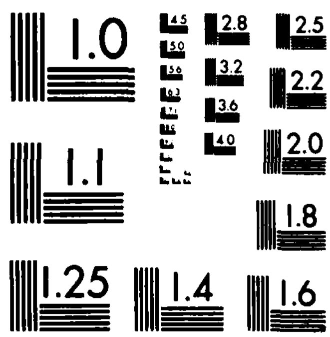

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Dyna-METRIC: Dynamic Multi-Echelon Technique for Recoverable Item Control

R. J. Hillestad

July 1982

Rand

PROJECT AIR FORCE

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The Dyna-METRIC model was developed to study and predict the readiness of groups of aircraft squadrons as determined by a major subset of logistics resources, namely, those associated with component repair and resupply. This report describes the mathematical approaches to modeling the effects of spare parts supply, component repair, and related processes on combat capability. It does not describe the implementation of any specific version of Dyna-METRIC. Section II reviews the time-dependent pipeline equations. Section III describes time-dependent stockage and component-repair measures of performance. Section IV combines these measures to give aircraft capability measures. Section V introduces the pipeline model for indentured components, while Section VI describes the pipeline equations for the time-dependent, multiple-echelon model. Section VII describes the optimization techniques for supply requirements, and Section VIII describes the approach for limited service facilities.
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Dyna-METRIC: Dynamic Multi-Echelon Technique for Recoverable Item Control

R. J. Hillestad

July 1982

A Project AIR FORCE report
prepared for the
United States Air Force

Rand
SANTA MONICA, CA. 90406

PREFACE

The Rand Corporation recently developed new analytic methods for studying the transient behavior of component-repair/inventory systems under time-dependent operational demands and logistics decisions like those that might be experienced in wartime (Hillestad and Carrillo, 1980). These methods have culminated in the Dyna-METRIC mathematical model described in this report.

Dyna-METRIC evolved through a series of projects that addressed policies for improving aircraft readiness and supportability. Initially, the dynamic queueing equations were developed and applied to the problem of determining appropriate levels of spare engines for the C-141 aircraft to support planned wartime surges in flying activity (Berman, Lippiatt, and Sims, 1978). Later, indentured component features and aircraft availability performance measures were added to study alternative logistics policies for carrier-based aircraft squadrons in the <u>Defense Resources Management Study</u> (Rice, 1979). Further features were added to enable the study of multiple bases of aircraft and their dependence on transportation of spare parts, in support of the Project AIR FORCE project "Responsive Intra-Theater Transportation System for Spare Parts." The model was then extended to include test equipment, limited service capacity, and test-equipment failure probability in the Project AIR FORCE project "Supporting Modern Tactical Avionics." Centralized repair and resupply capability features were added to allow the Dyna-METRIC model to be embedded in the experimental Combat Support Capability Management System being tested in the Pacific Air Forces (PACAF) under the Project AIR FORCE project "Combat Support Capability Management System." Most important, the models will be used in the Air Force's Worldwide Combat Supplies Management System (CSMS).[1]

Other initial implementations of the Dyna-METRIC model have been used by the Ogden Air Logistics Center, Air Force Logistics Command (AFLC), for studying USAF F-4 and F-16 aircraft readiness and supportability; by Headquarters AFLC for F100 engine evaluations; and by the Tactical Air Command to study the effect of several repair and supply strategies on the readiness and deployability of F-15 tactical squadrons.

This report is intended for users of the Dyna-METRIC computer model and for others who wish to understand the model's underlying mathematics. Complex theory has deliberately been avoided to facilitate this understanding. Hillestad and Carrillo (1980) describe some of the mathematics with greater theoretical rigor. The report does, however, assume in the reader a basic knowledge of calculus and probability theory.

The report describes the mathematical approaches to modeling the effects of spare parts supply, component repair, and related processes on combat capability. It does not describe the <u>implementation</u> of any specific version of Dyna-METRIC. The actual implementation is described in a forthcoming program description document (Pyles et al., 1982).

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[1] Actually there are several programmed versions of Dyna-METRIC. The version being embedded in CSMS is referred to as Dyna-METRIC, while all other versions are called RAMS (Rand Analytic Models of Support). In this report, the author refers to the entire generic class of models as Dyna-METRIC.

The publication of the report has been supported by the "Combat Support Capability Management System" study conducted within the Project AIR FORCE Resource Management Program.

SUMMARY

This report describes the Dyna-METRIC mathematical model for relating aircraft spare-parts supply levels and maintenance capability to material readiness of aircraft. A key characteristic of the model is its ability to deal with the dynamic or transient demands placed on component repair and inventory support caused by time variables in a scenario that includes sortie rates, mission changes, phased arrival of component repair resources, interruptions of transportation, and the like.

This has been accomplished by implementing a set of analytic mathematical equations describing the dynamic behavior of the component repair queueing system. The set also includes mathematical models of components and subcomponents (indentures) and multiple echelons of repair capability. From these equations the model can compute time-dependent inventory levels (spare parts requirements) and assess time-dependent mission readiness of the aircraft supported.

The text develops and describes the mathematical features of the model, providing insight into both its capabilities and the inherent assumptions.

ACKNOWLEDGMENTS

Many people have contributed to the ideas and approaches leading to this model. Thomas Lippiatt originally suggested looking at the effects of transients within the supply and component-repair system. Manuel Carrillo worked out some of the early dynamic queueing results and also suggested approaches for dealing with cannibalization and indentured components. Raymond Pyles designed the first implementations of the model, which eventually experienced widespread use at Rand and within the Air Force. Lieutenant Colonel Robert Tripp and Lieutenant Colonel Ronald Clarke tested many of the mathematical approaches using Air Force logistics data. Gail Halverson did most of the detailed programming to test and implement the model. Brian Leverich and Karen Isaacson refined some of the mathematical approaches and their implementation so as to make the model much more usable. Irving Cohen heavily influenced the design and use of Dyna-METRIC by orienting our work toward important logistics policy issues.

Glenn Gotz and Joan Buchanan have my sincere thanks for their many helpful comments and suggestions regarding the original manuscript.

Suzi Jackson and Dee Saenz also deserve thanks for their perseverance in putting this report together with its many mathematical equations and iterations.

CONTENTS

FIGURES

I. INTRODUCTION

Congress, the Department of Defense, and the services have each recognized the importance of being able to relate logistics resources and policies to readiness.[1] They have also recognized the difficulty of establishing that relationship. For example, the House Armed Services Committee Readiness Panel reports (1980) that "Current DOD readiness data and reports are not particularly helpful' to relating resources to future readiness . . ." The DOD Material Readiness Report (GAO, 1980) states,

> The Department of Defense (DOD) spends billions each year to maintain the readiness of its weapon systems but cannot accurately project how much readiness a dollar will buy or determine how much readiness is needed. . . . To date, DOD has made little progress in linking funding and material readiness and has not achieved an adequate material readiness report for the Congress. Its officials told GAO that the size and complexity of the problem has seriously hampered their attempts.

In describing the difficulty of measuring readiness, Charles W. Groover, Deputy Assistant Secretary of Defense (Program Integration) MRA&L, states that a major difficulty in managing readiness is, "the absence of analytical links between resources and readiness. We believe we have a pretty fair understanding of how the logistics system operates to support our combat weapon systems and equipment. . . . However, the specific functional relationships between resources applied and material

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[1] Readiness is defined as the state of availability and preparedness of resources with respect to their planned wartime mission. It differs from capability, a measure of the comparative effectiveness of different resources for accomplishing the same mission, and from effectiveness, which relates capability to its application against enemy forces.

readiness *resulting is incredibly complicated*." (Groover, 1977) The Air Force, in its Logistics Long Range Planning Guide (USAF, 1981), states that a major objective is to "develop a means to better identify and assess logistics requirements and capability, especially as these relate to execution of U.S. contingency plans."

Currently, readiness reporting by the services consists of rating the readiness of the separate groups of resources necessary to support a combat unit: supply, fuel, munitions, aircraft, operational personnel, maintenance personnel, etc. The group with the lowest rating defines the overall unit readiness. Unfortunately, the measures used for the different resources are not comparable and do not equally reflect the ability of the organization to perform its combat mission. For example, the supply system is measured by its ability to satisfy demands for spare parts, while pilots are measured by the number who are proficient and mission-ready. To achieve a balanced state of readiness, the services need techniques for describing readiness that use consistent measures and consider the interactions among resources.

The Dyna-METRIC model was developed to study and predict the readiness of groups of aircraft squadrons as determined by a major subset of logistics resources, namely, those associated with component repair and resupply. Thus, it attempts to solve two of the problems mentioned above: combining the influence of several types of support resources, and measuring their direct effect on mission readiness. Figure 1 illustrates the flow of aircraft components from the aircraft to various repair facilities to on-hand inventory, and from this inventory to the aircraft. The net objective of this system is to avoid the loss of aircraft mission capability due to shortages of correctly functioning components on the aircraft. Clearly, this objective can be met only if the local supply of those components exceeds the number of components tied up in various phases of repair or shipment. A snapshot in time would show some components awaiting repair, some being repaired, some in operable condition in inventory, some on the way to and from another echelon of repair, and some partially repaired but awaiting spare parts. Each of these states is a "pipeline" that contains some of the total inventory of components. When there are not enough spare components to cover each of these pipelines, "holes" will appear in aircraft; these "holes" may or may not affect the ability of the aircraft to perform a mission, depending on the mission essentiality of the missing component. The set ot resources considered in Dyna-METRIC includes the supply of spare aircraft components at bases and higher echelons, transportation resources, and the personnel and equipment used to perform repairs. The model links these resources to the unit's sortie capability. (Actually, transportation is represented by delays and interruptions, and personnel and equipment are represented by repair times and capacities of the repair process.)

The readiness of a unit to perform its mission depends on the availability of resources to support a highly dynamic flying program in the face of delays and interruptions of logistics support. Current methods for calculating required amounts of resources (such as spare parts and munitions) use "steady state" or time-averaging techniques that do not account explicitly for surges in sortie demands or variations in logistics support. Muckstadt (1980) has shown that spare-parts levels determined with such techniques can seriously understate the requirements during peak periods of activity. Berman, Lippiatt, and

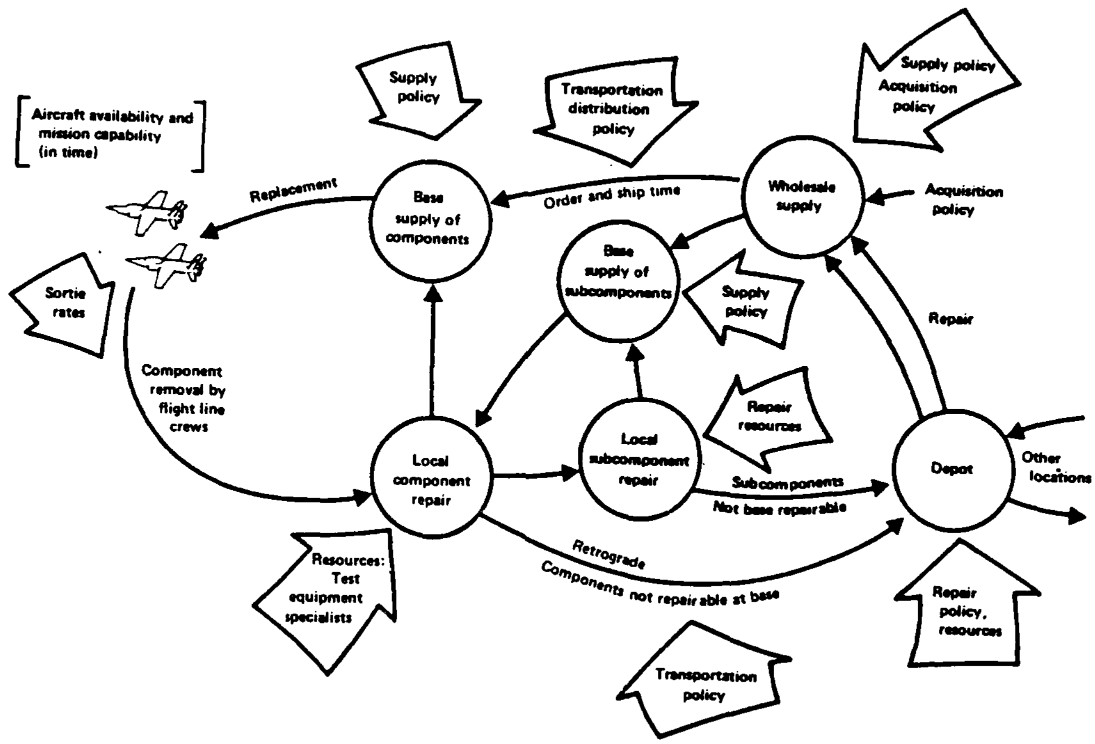

Fig. 1 -- Component repair and the readiness assessment problem

Sims (1978) have shown that logistics strategies (prepositioning spare engines, etc.) that are not apparent in the steady-state methods become more obvious when time dependence is considered explicitly. A key characteristic of the Dyna-METRIC model is its ability to deal directly with the transient demands placed on component repair and inventory support caused by dynamic parameters in a scenario (sortie rates, mission changes, phased arrival of component repair resources, interruptions of transportation, etc.). It does this by implementing a set of analytical mathematical equations describing the dynamic behavior of the component repair queueing systems, hence the term "Dyna" in the title of the model. It also includes mathematical models of components and subcomponents (indentures) and multiple echelons of repair capability. From these equations, it can compute time-dependent inventory levels (spare parts requirements) and assess the mission readiness of the aircraft supported. For these reasons, we borrowed the name METRIC from Sherbrooke (1968); this is an acronym for Multi-Echelon Technique for Recoverable Item Control, a mathematical model used to compute optimal inventory requirements for steady-state activity levels.

Hillestad and Carrillo (1980) give the theoretical development of the dynamic queueing equations that form the heart of Dyna-METRIC, as well as certain performance calculations related to component repair. This report reviews those results and describes how multiple echelons, multiple indentures, and requirements optimization can be incorporated in a dynamic model of logistics support. In describing the various aspects of the model below, we frequently state the classical steady-state approach prior to giving the time-dependent result; we do so to illustrate differences in the approaches. The implementation of Dyna-METRIC is described by Pyles et al. (1982).

Section II reviews the time-dependent pipeline equations. Section III describes time-dependent stockage and component-repair measures of performance. Section IV combines these measures to give aircraft capability measures. Section V introduces the pipeline model for indentured components, while Section VI describes the pipeline equations for the time-dependent, multiple-echelon model. Section VII describes the optimization techniques for supply requirements, and Section VIII describes the approach for limited service facilities.

The reader should be aware that Dyna-METRIC embodies many inherent assumptions regarding probability distributions, independence of variables, failure and repair of components, and allocation of shortages. Rather than list them out of context and without evaluative statements, we discuss the assumptions in the development of the model in the following sections.

II. TIME-DEPENDENT PIPELINES AND PROBABILITY DISTRIBUTIONS

Classical steady-state inventory theory provides a model that describes how many components will be in the various pipelines of a component-repair/inventory process when the component failure rates are driven by a probability distribution that is <u>not</u> time-dependent (that is, one that is stationary). Consider the case of a single type of component with an average daily failure rate, $\bar{d}$, and an average repair time, T. Steady-state theory (Palm, 1938), shows that, under the assumptions that the probability distribution of repair time is independent of the failure process, and that ample service capacity exists,[1] the average number of components in the repair pipeline will be[2]

$$\lambda_{ss} = \bar{d} \cdot T \tag{1}$$

Generally, aircraft components are assumed to fail at a rate based on the number of flying hours so that we can break $\bar{d}$, the average daily demand rate, into its constituents:

$$\bar{d} = \text{(average failures per flying hour)} \times \text{(flying hours/sortie)} \times \text{(average number of sorties per day per aircraft)} \times \text{(number of aircraft)} \times \text{(quantity of the components on the aircraft)} \times \text{(percentage of aircraft with the component)} . \tag{2}$$

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[1] The assumption of ample service capacity implies that no component awaits service and prevents the steady-state queue from becoming unbounded. The effect of limited service capacity is discussed further in Sec. VIII.

[2] The reader should be aware that we have changed the usual meaning of λ in Little's Theorem, which states the same result as $L = \lambda W$. Thus $L = \lambda_{ss}$, $\lambda = \bar{d}$, $W = T$.

With the further assumption that d has a Poisson probability distribution the average number of components in the steady-state pipeline, λ_{ss}, actually represents the mean value of a Poisson distribution as well. By Palm's Theorem, the probability that there are k components in the pipeline at any point in time is given by

$$P(k \text{ in pipeline}) = \frac{\lambda_{ss}^{k} e^{-\lambda_{ss}}}{k!} \qquad (3)$$

The Poisson distribution arises when the mean time (in flying hours) between failure of components follows an exponential probability distribution, and is frequently borne out in data analysis of failure rates. There are also strong theoretical reasons to expect Poisson failures.[3]

Time-dependent flying scenarios cannot be considered correctly with this approach. It is necessary to use the average number of aircraft and average number of sorties per day per aircraft rather than the time-dependent values. In addition, time-dependent repair capability cannot be considered because the average repair time, T, must be used in the steady-state model.

The dynamic model used in Dyna-METRIC relieves these assumptions. In this case we let the daily demand rate, d(t), be a function of time so that

d(t) = (failures per flying hour) × (flying hours/sortie at time t) × (number of sorties per day per aircraft at time t) × (number of aircraft at time t) × (quantity of the component on the aircraft) × (percentage of aircraft with the component) . (4)

[3] See, e.g., Samuel Karlin and H. M. Taylor, A First Course in Stochastic Processes, 2d ed., Academic Press, New York, 1975, pp. 221-228.

This expression for d(t) includes several variables that might be expected to change. For example, the number of aircraft can change in time according to the time sequence of deployment or because of aircraft attrition. The number of sorties per day per aircraft changes as a result of programmed changes in flying rates, and the flying hours per sortie change as missions change.

In place of a constant average repair time, T, the dynamic model uses the probability that a component entering repair at time s is still in repair at time t. This probability function, $\bar{F}(t,s)$, is called the <u>repair function</u>. It is defined by

$$\bar{F}(t,s) = \text{Prob} \begin{Bmatrix} \text{Component entering at s is} \\ \underline{\text{still}} \text{ in repair at t} \end{Bmatrix}$$

$$= \text{Prob} \begin{Bmatrix} \text{Repair time} > \text{t-s when} \\ \text{started at s} \end{Bmatrix}$$

A few illustrations will show that this function is relatively simple to obtain for most uses, but also has considerable power to represent certain time dependencies in component repair. (It is assumed to be independent of the failure process, however.)

i. Constant or fixed repair time, T.

$$\bar{F}(t,s) = \begin{cases} 1 \text{ if } t - s < T \\ 0 \text{ if } t - s \geq T \end{cases} \quad (5)$$

(That is, if the time between component arrival, s, for repair and the current time, t, is less than the constant repair time, T, then the probability that it will still be in repair is 1; otherwise it is 0).

ii. Exponentially distributed repair time with average T.

$$\bar{F}(t,s) = e^{-\frac{t-s}{T}} \tag{6}$$

iii. No repair capability until time, τ, with exponentially distributed repair time after τ.

$$\bar{F}(t,s) = \begin{cases} 1 \text{ if } t < \tau \\ e^{-\frac{t-\tau}{T}} & \text{if } s \leq \tau \leq t \\ e^{-\frac{t-s}{T}} & \text{if } \tau < s \leq t \end{cases} \tag{7}$$

iv. Fixed transportation lag, S, with exponentially distributed repair time after transportation occurs.

$$\bar{F}(t,s) = \begin{cases} 1 & \text{if } t - s < S \\ e^{-\frac{t-(s+S)}{T}} & \text{if } t - s \geq S \end{cases} \tag{8}$$

v. Exponentially distributed repair time which changes at time, τ, (from T_1 to T_2 average repair time)

$$\bar{F}(t,s) = \begin{cases} e^{-\frac{t-s}{T_1}} & \text{if } t \leq \tau \\ e^{-\frac{t-\tau}{T_2}} \cdot e^{-\frac{\tau-s}{T_1}} & \text{if } s < \tau < t \\ e^{-\frac{t-s}{T_2}} & \text{if } \tau \leq s < t \end{cases} \tag{9}$$

Of course, many other repair functions could be modeled provided they are independent of the demand function. Section VIII shows how dependent repair functions may be approximated. We will now describe how Dyna-METRIC combines the repair and demand functions to determine the average number of parts in the pipeline. Consider only those components that arrived in an interval of time, Δs, centered at time s. The number expected in the repair pipeline at time t will then be given by

$$\Delta\lambda(t,s) = d(s) \times \bar{F}(t,s) \times \Delta s \ , \tag{10}$$

where $\Delta\lambda(t,s)$ = expected number of components in the repair pipeline at time t that arrived during the interval around s;

$d(s)$ = daily failure rate at time s;

$\bar{F}(t,s)$ = probability of component not out of repair by time t; and

Δs = interval of time centered at s.

If we assume that the number of failures arriving in the interval Δs is independent of the number of failures arriving in similar intervals centered at other times other than s[4] and that the repair probability function is independent of the probability distribution generating the demand rate, we can sum the contributions of all intervals to obtain

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[4] This is called the independent increment assumption and is true for demands governed by a Poisson process. Actually, there are reasons that component repair demands <u>may</u> <u>not</u> have independent increments. If aircraft are decreasing in availability because of component failures and there is a resulting decrease in sorties flown, then the later sorties (and hence component failures) are a function of previous component removals. This will cause the result above to be an overstatement of the number of components in the pipeline. For most users of Dyna-METRIC this will not seriously affect the answer; but for some cases with serious shortages of supply resources, a second iteration with reduced sorties based on the previous iteration will give a more accurate answer. Generally, the <u>overstatement</u> of pipelines that may occur will give conservative answers to stockage requirements and to capability with a fixed level of stock.

$$\lambda(t) = \sum_{s \leq t} \Delta\lambda(t,s)$$

$$= \sum_{s \leq t} d(s) \cdot \bar{F}(t,s)\Delta s \tag{11}$$

If we make ds arbitrarily small, we arrive at the integral expression for $\lambda(t)$.

$$\lambda(t) = \int_0^t d(s)\bar{F}(t,s)ds \tag{12}$$

This integral represents the average number of components in the repair pipeline at time t. Hillestad and Carrillo (1980) show that, with the additional assumption that the component failure probability distribution is Poisson, $\lambda(t)$ is the mean of a nonhomogeneous (time varying) Poisson process. That is, the probability of k components in repair at time t is

$$P(k) = \frac{\lambda(t)^k \, e^{-\lambda(t)}}{k!} \tag{13}$$

where

$$\lambda(t) = \int_0^t d(s)\bar{F}(s,t)ds \ . \tag{14}$$

The integration from 0 to t assumes no flying before 0 (or at least $\bar{F}(s,t) = 0$ for $s < 0$). Superposition allows consideration of flying during this period. Peacetime or steady-state pipeline quantity preceding $t = 0$ can be considered by computing $\lambda(0) \cdot \overline{F}(t,0)$, the quantity of the pipeline remaining at time t. Under other assumptions it is possible for $\lambda(t)$ to represent the mean value of other probability distributions. Compound Poisson distributions also satisfy the independent increment assumption and lead to pipeline distributions such as the negative binomial distribution given by

$$P(k) = \frac{(r + k - 1)!}{(r - 1)!k!} \cdot \frac{p^k}{q^{r+k}} \tag{15}$$

where q = variance to mean ration $(q > 1)$

$$r = \frac{\lambda(t)}{q - 1} \tag{16}$$

$$p = q - 1 \tag{17}$$

and $\lambda(t)$ is given by the equation shown above. This distribution can arise when groups of failures occur at instants of time (e.g., due to the arrival of groups of aircraft) and the number of instants of failure (number of batches of failures) in a given time interval is governed by a nonhomogeneous Poisson distribution. When the distribution of the number of failures in any group (at any instant of time) is given by a logarithmic distribution, the resulting distribution of the pipeline's quantity is negative binomial. (The Poisson distribution allows only <u>one</u> failure at each instant.) The additional information necessary to use this distribution is q, the variance-to-mean ratio. This must be determined by statistical estimation of the variance of historical failure data. (The Poisson distribution has a variance-to-mean ratio of 1 and does not need this variable.)

An important aspect of estimating pipeline quantities is the ability to separate pipelines with dissimilar demand rates and repair functions for ease of calculation, and then to sum the result to obtain the average total quantity of components tied up in all of the pipelines. This is easiest when the components flowing into each pipeline arise from a common source distribution. For example, suppose that demands for component repairs arise at an aircraft squadron based on the flying activity of the squadron, and that there is a probability p that a failed component can be fixed locally and 1 - p that it must be shipped elsewhere to be repaired. Further, assume that the local repair has probability function $\bar{F}_1(s,t)$ and that remote repair has probability function $\bar{F}_2(s,t)$ (including transportation time). Then we have that

$$\lambda(t) = \int_0^t d(s)[p\bar{F}_1(s,t) + (1 - p)\bar{F}_2(s,t)]ds \tag{18}$$

or that

$$\lambda_1(t) = \int_0^t d(s)p\bar{F}_1(s,t)ds \tag{19}$$

$$\lambda_2(t) = \int_0^t d(s)(1 - p)\bar{F}_2(s,t)ds \tag{20}$$

$$\lambda(t) = \lambda_1(t) + \lambda_2(t) . \tag{21}$$

Thus, it is possible to separate and recombine the average values for the various component repair pipelines in the dynamic model just as in the steady-state model. This is true regardless of the probability distributions of different parts of the pipeline. However, when the distributions are of different form or have differing variance-to-mean ratios, the resulting total distributions can be complicated. When the distributions in each pipeline are independent, we can determine the resulting distribution by the more complex process of convolution.

This section will conclude with an illustration of the dynamic pipeline calculation and time-dependent probabilities. Consider an aircraft squadron at a single location with:

$N_a(s) \equiv$ aircraft at time s,

$D(s) \equiv$ sorties per aircraft demanded at time s, and

$FH(s) \equiv$ flying hours per sortie at time s.

A component on the aircraft has the following characteristics:

$m \equiv$ failures per flying hour;

$q \equiv$ quantity per aircraft.

The demand function for this squadron for this component is then:

$$d(s) = N_a(s) \cdot D(s) \cdot FH(s) \cdot m \cdot q \tag{22}$$

Each failure of the component has a probability, p, of being repaired locally and 1 - p of being repaired remotely. If it is repaired locally, then it can be repaired in an exponentially distributed time with mean T^1 except for the first τ_R days during which no repair capability exists. If it is repaired remotely, an order is placed for a new component, which is immediately shipped and arrives after a fixed transportation delay T^2. The number of aircraft and flying hours per sortie will be assumed constant in this example so that NA(s) = Na and FH(s) = FH.

The demanded flying program will have one step at time zero to D_1 sorties per day, and a second step to a lower number of sorties D_2 per day at τ_D. (We will assume that $\tau_D > \tau_R$, that $T^1 > T^2$, and that $T^2 > \tau_D$.) The pipeline integral for the component is then

$$\lambda(t) = \int_0^t d(s)[p\bar{F}^1(s,t) + (1 - p)\bar{F}^2(s,t)]ds , \tag{23}$$

where

$$d(s) = \begin{cases} N_a \cdot D_1 \cdot FH \cdot m \cdot q = d^1 & s < \tau_D \\ N_a \cdot D_2 \cdot FH \cdot m \cdot q = d^2 & s \geq \tau_D \end{cases} \tag{24}$$

where

$$\bar{F}^1(s,t) = \begin{cases} 1 & t < \tau_R \\ e^{-\frac{t-\tau_R}{T^1}} & s < \tau_R \leq t \\ e^{-\frac{t-s}{T^1}} & \tau_R \leq s \leq t \end{cases} \tag{25}$$

$$\bar{F}^2(s,t) = \begin{cases} 1 & \text{if } t - s < T^2 \\ & \\ 0 & \text{if } t - s \geq T^2 \end{cases} \tag{26}$$

Let $\lambda^1(t)$ be the average local repair pipeline quantity at time t, and let $\lambda^2(t)$ be the average remote repair pipeline at time t. Then,

$$\lambda^1(t) = \int_0^t d(s)\ p\bar{F}^1(s,t)ds \tag{27}$$

and

$$\lambda^2(t) = \int_0^t d(s)(1 - p)\bar{F}^2(s,t)ds\ . \tag{28}$$

These integrals further separate into

$$
\lambda^1(t) = \begin{cases}
\int_0^t d^1 \cdot p \, ds & t < \tau_R \\[2ex]
\int_0^{\tau_R} d^1 \cdot p \, e^{-\frac{t-\tau_R}{T^1}} ds + \int_{\tau_R}^{t} d^1 p \, e^{-\frac{t-s}{T^1}} ds & \tau_R \le t < \tau_D \\[2ex]
\int_0^{\tau_R} d^1 \cdot p \, e^{-\frac{t-\tau_R}{T^1}} ds + \int_{\tau_R}^{\tau_D} d^1 p \, e^{-\frac{t-s}{T^1}} ds + \int_{\tau_D}^{t} d^2 p \, e^{-\frac{t-s}{T^1}} ds & \tau_D \le t
\end{cases}
\tag{29}
$$

$$\lambda^2(t) = \begin{cases} \displaystyle\int_0^t d^1(1 - p)ds & t < \tau_D \\[2ex] \displaystyle\int_0^{\tau_D} d(1 - p)ds + \int_{\tau_D}^t d^2(1 - p)ds & \tau_D \leq t < T^2 \\[2ex] \displaystyle\int_{t-T^2}^{\tau_D} d(1 - p)ds & \\[2ex] \displaystyle\quad + \int_{\tau_D}^{T^2} d^2(1 - p)ds & \\[2ex] \displaystyle\quad + \int_{T^2}^{t} d^2(1 - p_1)ds & T^2 \leq t < \tau_D + T^2 \\[2ex] \displaystyle\int_{t-T_2}^{T_2} d^2(1 - p)ds & \tau_D + T^2 \leq t \end{cases} \tag{30}$$

Performing the integration we obtain the average pipeline quantities given by

$$\lambda^1(t) = \begin{cases} d^1 \cdot p \cdot t & t < \tau_R \\ d^1 \cdot p \cdot \tau_R \, e^{-\frac{t-\tau_R}{T^1}} + d^1 p \, T^1 \left(1 - e^{-\frac{t-\tau_R}{T^1}}\right) & \tau_R \le t \le \tau_D \\ d^1 p \, \tau_R \, e^{-\frac{t-\tau_R}{T^1}} + d^1 p \, T^1 \left(e^{-\frac{t-\tau_D}{T^1}} - e^{-\frac{t-\tau_R}{T^1}}\right) & \\ \quad + d^2 p T^1 \left(1 - e^{-\frac{t-\tau_D}{T^1}}\right) & \tau_D \le t \end{cases} \tag{31}$$

and

$$\lambda^2(t) = \begin{cases} d^1(1 - p)t & t < \tau_D \\ d^1(1 - p)\tau_D + d^2(1 - p)(t - \tau_D) & \tau_D \le t < T^2 \\ d^1(1 - p)(\tau_D - t + T^2) & \\ \quad + d^2(1 - p)(t - \tau_D) & T^2 \le t < \tau_D + T^2 \\ d^2(1 - p)T^2 & \tau_D + T^2 \le t \end{cases} \tag{32}$$

Figure 2 illustrates these results.

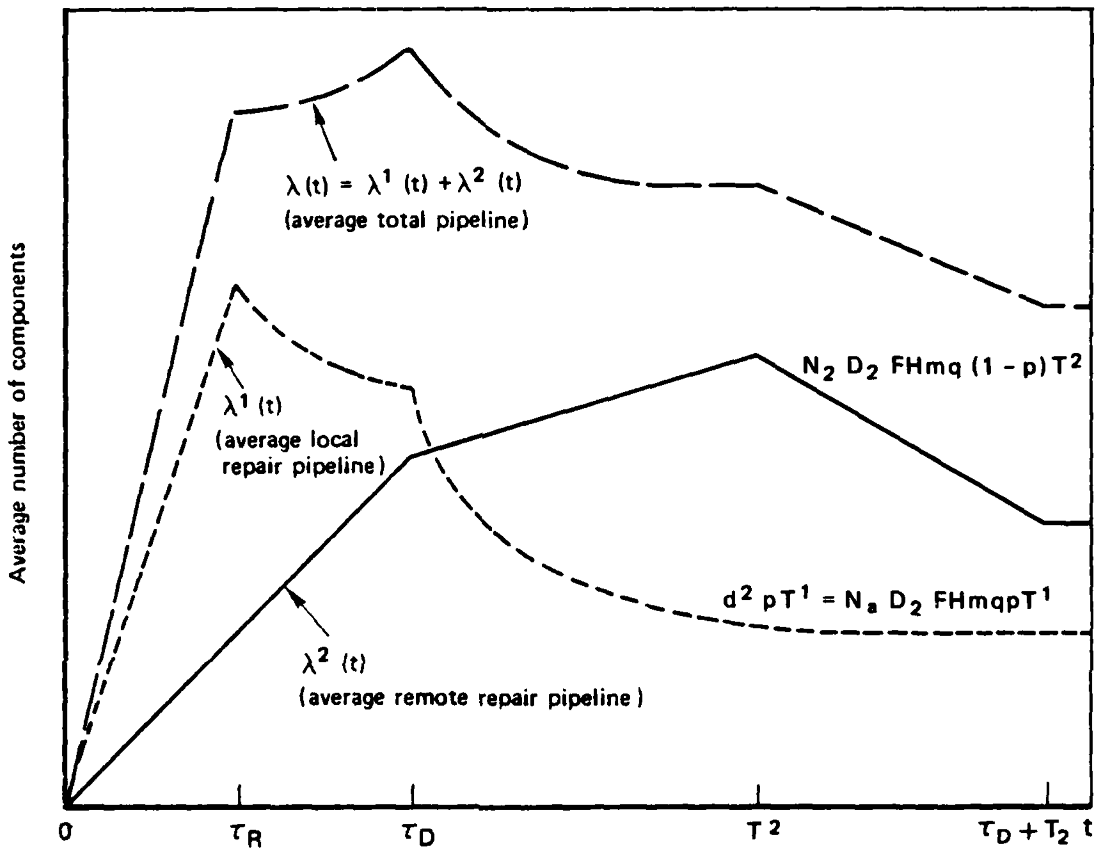

Fig. 2—Illustration of average pipeline calculations

The addition of the probability distribution allows us to determine the probability of various pipeline quantities. For example, if the demand distribution is Poisson, the pipeline distribution will be Poisson, as described earlier. We can determine, for this example, the probability that no more than K components will be in the local repair pipeline using the following calculations:

$$p(\text{no. in local pipeline} \leq K) = \sum_{k=0}^{K} \frac{\lambda_1(t)^k e^{-\lambda_1(t)}}{k!} \qquad (33)$$

This is illustrated in Fig. 3.

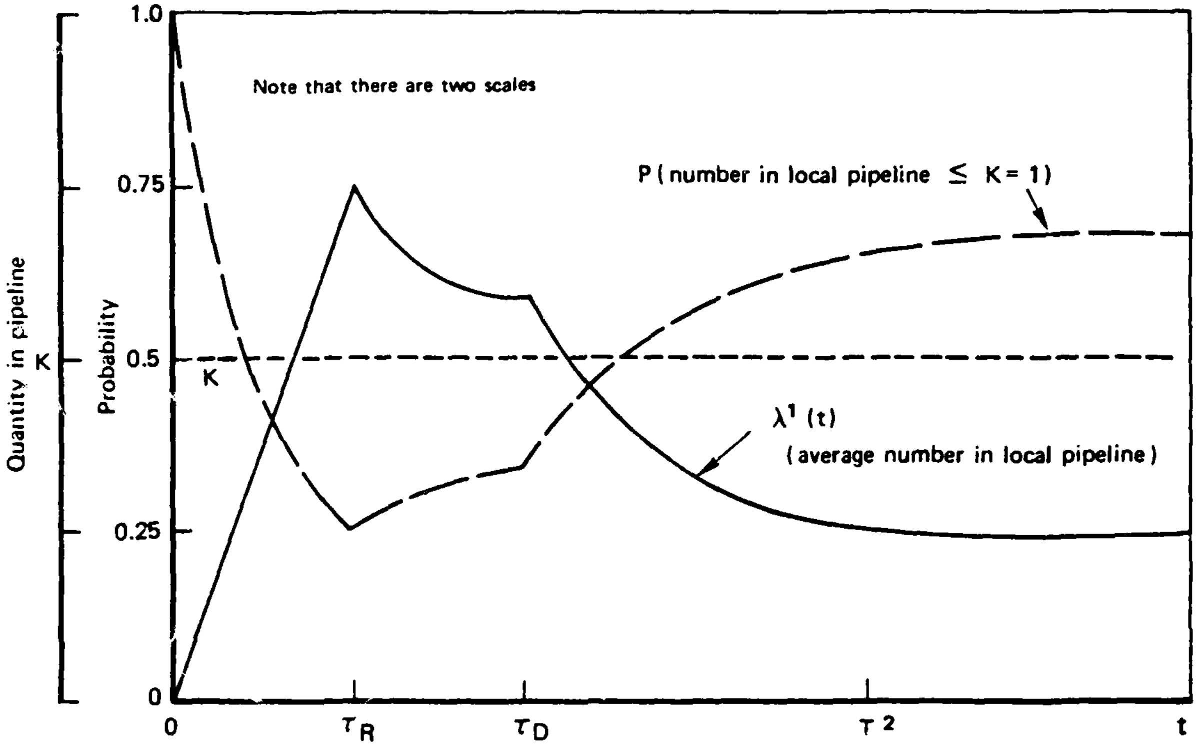

Fig. 3—The probability that no more than K components are in local repair

Another important use of the probability distribution is to determine confidence intervals. Suppose, in the example, we wish to determine the number of components in local repair that we are quite sure will not be exceeded. We might quantify this by looking for the number that will not be exceeded with 90 percent probability. That is, we desire 90 percent confidence (or .9 probability) that the number will not be exceeded.

Let L(t) be the number of components that satisfies this confidence. The value of L(t) can be determined by summing the Poisson terms until they equal or exceed .9 at each time instant of interest. Thus, we find L(t) by solving the following equation for the smallest value of L(t):

$$.9 = \sum_{k=0}^{L(t)} \frac{\lambda_1(t)^k e^{-\lambda_1(t)}}{k!}$$

This is illustrated in Fig. 4.

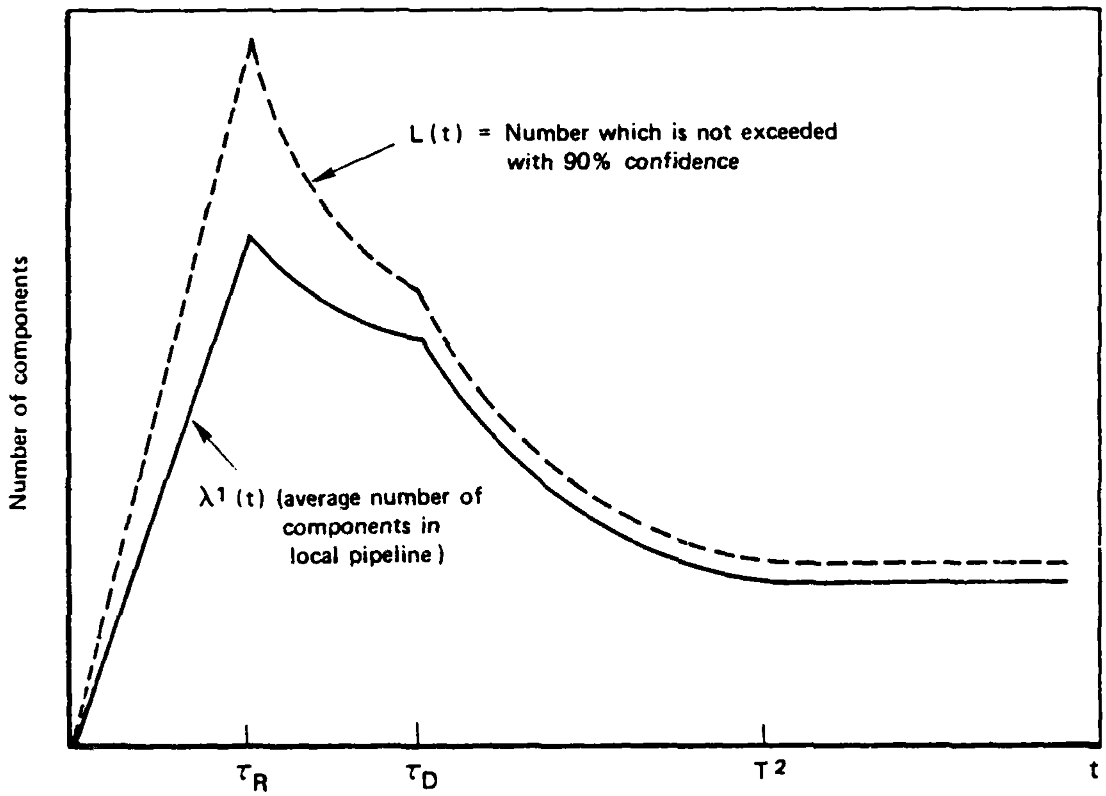

Fig. 4—Illustration of confidence level

III. TIME-DEPENDENT COMPONENT PERFORMANCE MEASURES

Section II described the calculation of time-dependent average component pipeline quantities and their associated probability distributions. Combining these pipeline quantities with the supply levels at the same instant of time allows the determination of various measures describing the availability or shortages of individual components at the aircraft. These are the supply or component-repair measures used by the services to describe supply system performance, except that in this case time dependence is included. They are also similar to the measures used by the services to determine the required spare-parts levels. These measures do not completely describe aircraft mission readiness, since they do not describe the combined loss of capability that shortages of groups of components inflict on the aircraft when cannibalization (consolidation of shortages) and mission essentiality are considered. They do form a basis for higher-level measures, and they are also useful in identifying problem components once overall system performance has been determined to be inadequate. The next section will deal with the joint effect of the components.

The component performance measures are determined for a given supply level, S(t). This level is provided to protect the aircraft squadron from shortages due to components in repair or on order. When the number of components in repair plus on order at time t exceeds S(t), then the system is said to be in a "backorder" state. When components are backordered, the aircraft will be missing components, possibly degrading the mission or sortie capability.

The component measures typically computed by the Dyna-METRIC model are:

$R(t) \equiv$ Ready rate at time t--the probability that an item observed at time t has no backorders.

$FR(t) \equiv$ Fill rate at time t--the probability that a demand at time t can be filled immediately from stock on hand.

$EB(t) \equiv$ Expected back orders--the average number of shortages of a component at time t.

$VBO(t) \equiv$ Variance of the backorders, a measure of the random variation of backorders.

$DT(t) \equiv$ Average cumulative demands by time t.

The last measure is derived from the daily demand rate, d(s), as:

$$DT(t) = \int_0^t d(s)ds \ . \tag{1}$$

The remaining measures use the average pipeline quantity, $\lambda(t)$, the stock level, S(t), and the probability distribution $P(k/\lambda(t))$[1] chosen from the Poisson distribution (variance-to-mean ratio = 1.0) or the negative binomial (variance-to-mean ratio > 1.0) distribution.

The ready rate is given by

$$R(t) = \sum_{k=0}^{S(t)} P(k/\lambda(t)) \tag{2}$$

---

[1] The notation $P(k/\lambda(t))$ is introduced here to indicate the time dependence and mean of the probability distribution. It is, however, the same probability as described by P(k) in Sec. II.

The interpretation of this expression is as follows: $P(k/\lambda(t))$ is the probability of there being exactly k components in the pipeline when the average number is $\lambda(t)$. $P(0/\lambda(t))$ is the probability of zero in the pipeline. The sum of $P(0/\lambda(t))$, $P(1/\lambda(t))$, ..., up to $P(S/\lambda(t))$ is the probability that S or fewer components will be in the pipeline. This is the probability that the pipeline will contain no more than the stock level and is therefore the probability that there will be no backorders.

The <u>fill rate</u> is given by

$$FR(t) = \sum_{k=0}^{S(t)-1} P(k/\lambda(t)) \; . \tag{3}$$

The interpretation of this expression is as follows: The fill rate is the probability that a component will be available when a demand is placed. It is therefore the probability that demands have left at least 1 component available and is therefore the sum of the probabilities of demands <u>less</u> than the stock level.

<u>Expected backorders</u> are given by

$$\begin{aligned} EB(t) &= \sum_{k=S(t)+1}^{\infty} (k - S(t) P(k/\lambda(t)) \\ &= \lambda(t) - S(t) + \sum_{k=0}^{S(t)} (S(t) - k) P(k/\lambda(t)) \; . \end{aligned} \tag{4}$$

The interpretation of the first expression is as follows: There are no backorders if stock equals or exceeds the demand. Therefore, for k less than or equal to $S(t)$, the backorders are zero. For k greater than $S(t)$, the backorders are merely $(k - S(t))$. The probability of any demand level, k, is $P(k/\lambda(t))$, and the expected value of the backorders is merely the product of the various values the backorders can take on times the probability of a demand at that given value. The second expression for the expected value is more useful computationally; it is obtained by using the facts that

$$\sum_{k=0}^{\infty} kP(k/\lambda(t)) = \lambda(t) \quad \text{and} \tag{5}$$

$$\sum_{k=0}^{\infty} S(t)P(k/\lambda(t)) = S(t) . \tag{6}$$

The <u>variance</u> <u>in</u> <u>backorders</u> is given by

$$VB(t) = \sum_{k=S(t)+1}^{\infty} [k - S(t)]^2 \, P(k/\lambda(t)) - [EB(t)]^2 . \tag{7}$$

For the Poisson Distribution this is given also by the computationally tractable expression,

$$\begin{aligned} VB(t) &= \lambda(t) + [\lambda(t) - S(t)]^2 - [EB(t)]^2 \\ &\quad - \sum_{k=0}^{S(t)} [k - S(t)]^2 \, P(k/\lambda(t)) . \end{aligned} \tag{8}$$

This expression was derived using the previous results and the fact that, for the Poisson distribution,

$$\sum_{k=0}^{\infty}(k - \lambda(t))^2 \; P(k/\lambda(t)) = \lambda(t) \; . \tag{9}$$

These measures can be computed for each component for each location and time of interest. They provide a detailed picture of the component's status. Figure 5 illustrates the component measures for a case in which the average pipeline quantity undergoes a buildup in time.

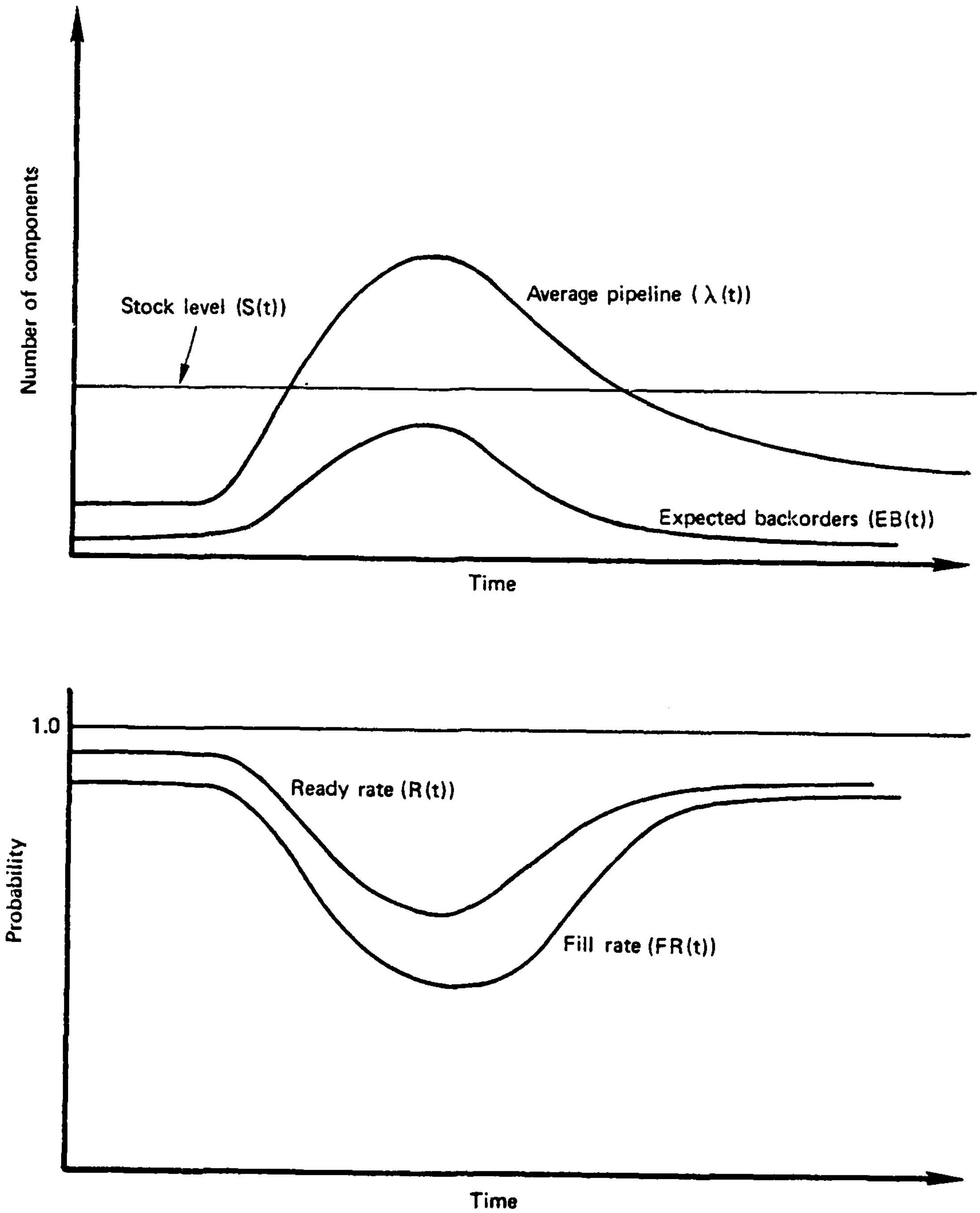

Fig. 5--Illustration of component measures

IV. TIME-DEPENDENT SYSTEM PERFORMANCE MEASURES

Dyna-METRIC also forecasts the effect of component shortages (due to shortage of spares, inadequate component-repair capability, etc.) on the number of mission-capable aircraft and consequent ability to generate missions. These measures show the combined effect of the component shortages on aircraft availability, both with and without "cannibalization" to minimize the number of aircraft with "holes." Under cannibalization, the model assumes the ability to instantly consolidate shortages onto the smallest number of airframes. In actuality this would be done only when the aircraft are needed. Thus, this measure may overstate the number of aircraft that might be available in unstressed conditions (when all aircraft are not needed). On the other hand, the measures for conditions of no cannibalization may understate aircraft availability, because some cannibalization almost always occurs in practice. The true answer probably lies somewhere in between these bounds, closer to the full cannibalization values in stressful conditions and closer to the no-cannibalization numbers under less stressful conditions. The average total number of backorders is a useful system measure that describes the total number of holes in aircraft.

Mission essentiality of components can be accommodated by sets of indicators for each component showing whether or not the component is required for a given, finite set of missions. Shortages of components that are not considered mission-essential at a point in time are then not included in determining the aircraft performance measures.

The following set of system performance measure equations is repeated from Hillestad and Carrillo (1980):

1. Average number of systems NMC (not mission capable) without cannibalization, EN(t).

This measure gives the average combined effect of recoverable item shortages on the aircraft that those items support. Given that NA(t) represents the number of aircraft that are supported at time t, and that there are N types of recoverable items required on each aircraft, assume that the shortages of any one of these items will make the system nonoperational. Also, assume that shortages of components cannot be consolidated among the systems.

The probability that an arbitrary aircraft is missing component type i when there are k shortages of i across the fleet of NA(t) aircraft is[1]

$$\frac{k}{NA(t)} \tag{1}$$

and therefore, the probability that an arbitrary aircraft has a shortage of item i is given by summing across the possible values of $k = \ell - s_i(t)$ times the probability they occur:

$$\sum_{\ell=S_i(t)+1}^{\infty} \frac{(\ell - S_i(t))}{NA(t)} \; P(\ell/\lambda_i(t)) = \frac{EB_i(t)}{NA(t)} , \tag{2}$$

where EBi(t) is the expected backorders on component i. Assuming that failures are independent, the probability that an aircraft is non-mission-capable due to shortages of some item is

$$1 - \prod_{i=1}^{N} \left(1 - \frac{EB_i(t)}{NA(t)}\right) . \tag{3}$$

[1] This assumes that there is at most 1 of each component of type i on any aircraft. The modification for multiple occurrences on an aircraft will be discussed shortly.

And, finally, the expected number of non-mission-capable aircraft at time t is

$$EN(t) = NA(t)\left[1 - \prod_{i=1}^{N}\left(1 - \frac{EB_i(t)}{NA(t)}\right)\right]. \tag{4}$$

This derivation required the assumption that only one component of a given type was on each aircraft. If this is not the case, let Q_i be the quantity of item i per system. If no consolidation of shortages occurs, then

$$EN(t) = NA(t)\left[1 - \prod_{i=1}^{N} \Pr\text{ (acft have no shortages of i)}\right]$$

$$= NA(t)\left[1 - \prod_{i=1}^{N} \sum_{y=0}^{NA(t)\cdot Q_i} \Pr\text{ (acft have no shortages of i when y shortages of part i exist in the squadron)}\right]$$

(5)

$$= NA(t)\left[1 - \prod_{i=1}^{N} \sum_{y=0}^{NA(t)\cdot Q_i} \frac{\binom{Q_i NA(t)-y}{Q_i}}{\binom{Q_i\cdot NA(t)}{Q_i}} PB_i(y)\right]$$

where PBi(y) is the probability that item i has y shortages at time t (the probability distribution of backorders of component i).

$$PB_i(y) = \begin{cases} \displaystyle\sum_{k=0}^{S_i(t)} P(k/\lambda_i(t)) & y = 0 \\ P(k + S(t)/\lambda_i(t)) & y > 0 \end{cases} \tag{6}$$

2. Average number of aircraft NMC (non-mission-capable) with an instantaneous cannibalization policy, $EN_c(t)$.

In this measure it is assumed that shortages of all components are consolidated to make the smallest number of aircraft non-mission-capable. Let $P^i(j)$ be the probability that shortages of the ith item are less than or equal to j. Then

$$P^i(j) = \sum_{k=0}^{S_i(t)+j} P(k/\lambda_i(t)) \; . \tag{7}$$

Let P(j) be the probability that the number of non-mission-capable aircraft is less than or equal to j. Then

$$P(j) = \prod_{i=1}^{N} P^i(j) \; . \tag{8}$$

If there is more than one item of a type on each aircraft, we again employ Q_i as the number of item i per aircraft and obtain

$$P(j) = \prod_{i=1}^{N} P^i(Q_i \cdot j) \; . \tag{9}$$

This measure is important by itself as well as being a step toward obtaining the average number of NMC aircraft under full cannibalization. It is used as a performance constraint equation in the optimization procedure in Sec. VIII. Ebeling (1978) refers to this distribution as the "Operational Ready Rate," and argues that it is the most operationally oriented performance measure among those commonly considered for a spare-parts objective function.

The expected number of NMC aircraft with full cannibalization is then (using the fact that the expected value of a nonnegative distribution is equal to the sum of the complement of that distribution):

$$EN_c(t) = \sum_{j=0}^{NA(t)-1} [1 - P(j)] \ . \tag{10}$$

The NMC distribution function with full cannibalization is

$$PN_j(t) = P(j) - P(j - 1) \ , \tag{11}$$

which then can be used to give the variance in the number of NMC aircraft,

$$VN(t) = \left[\sum_{j=1}^{NA(t)} j^2 \ PN_j(t)\right] - EN_c^2(t) \tag{12}$$

3. <u>Total expected backorders</u>.

This measure is the sum of the individual component backorders, $BO_i(t)$. Thus,

$$EB(t) = \sum_{i=1}^{N} EB_i(t) \ . \tag{13}$$

This is commonly used as an optimization objective or constraint in determining supply levels. It is a very conservative estimate of operational performance, as can be seen by expanding the expected NMC expression with no cannibalization and assuming $Q_i = 1$ for all i.

$$EN(t) = NA(t)\left[1 - \left(1 - \sum_{i=1}^{N} \frac{EB_i(t)}{NA(t)} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{EB_i(t)}{NA(t)} \cdot \frac{EB_j(t)}{NA(t)} \cdots\right)\right] \tag{14}$$

when

$$\frac{EB_i(t)}{NA(t)} << 1.0 \ , \tag{15}$$

when

$$EN(t) \approx \sum_{i=1}^{N} EB_i(t) = EB(t) \tag{16}$$

Thus, if $Q_i = 1$ for all components and $E_i(t)$ is very small (relative to the number of aircraft) for each component, then EB(t) is an approximation to the expected number of non-mission-capable aircraft with no cannibalization and will always overstate EN(t).

4. NMC with partial cannibalization.

Here we assume that some components are relatively easy to cannibalize, and that some are so difficult to remove or install that it is not desirable to cannibalize them. Let

$I_c = \{ i \mid \text{component i is cannibalizable} \}$

$I_n = \{ i \mid \text{component i is not cannibalizable} \}$

We first compute the probability of exactly k NMC aircraft due to items in the set I_c. This is merely the $PN_j(t)$ derived earlier using $P(j)$, where $P(j)$ is computed using the i in I_c. We will denote this probability $PN_j^c(t)$. The probability that an arbitrary aircraft is nonoperational, considering only shortages of cannibalizable items, is then

$$\sum_{j=1}^{NA(t)} \frac{j}{NA(t)} PN_j^c(t) = \frac{EN^c(t)}{NA(t)} . \tag{17}$$

The probability that an arbitrary system is operational after shortages of noncannibalizable components only is

$$\prod_{i \in I_n} \left(1 - \frac{EB_i(t)}{NA(t)}\right), \tag{18}$$

assuming that Q_i, the quantity of item i per system, is one for each part, i, in the set I_n. The probability that an arbitrary aircraft is not operational, assuming independence of demands and that the cannibalization of the items in I_c takes place with no information of the failed items belonging to I_n, is

$$1 - \left[1 - \frac{EN^c(t)}{NA(t)}\right] \left[\prod_{i \in I_n} (1 - \frac{EB_i(t)}{NA(t)}\right] ; \tag{19}$$

and the expected number of nonoperational aircraft, given partial cannibalization and given NA(t), total aircraft, is

$$EN^{p}(t) = NA(t)\left(1 - \left[1 - \frac{EN^{c}(t)}{NA(t)}\right]\left[\prod_{i \varepsilon I_n}\left(1 - \frac{EB_i(t)}{NA(t)}\right)\right]\right). \qquad (20)$$

$EN^{p}(t)$ is an overestimation of the expected NMC if information describing which noncannibalizable items had failed is used (since, in this case, the strategy might be to move "holes" due to cannibalizable items to those aircraft with holes due to noncannibalizable items. This was not considered in the derivation).

5. <u>Probability of meeting aircraft missions</u>.

This performance measure is for the situation in which it is desirable to have enough operational aircraft to perform a certain demanded level of activity. Let B(t) be the maximum number of allowable NMC aircraft that still permits the missions to be met, and let D(t) be the number of missions at time t. Let r(t) be the maximum number of sorties per unit time achievable by a single mission-capable aircraft. Note that

$$B(t) = NA(t) - \frac{D(t)}{r(t)}, \qquad (21)$$

where $\lceil x \rceil$ is the ceiling of x, that is, the smallest integer so that $\lceil x \rceil \geq x$, and $D(t) \geq r(t)$. NA(t) is assumed to be an integer. The probability of meeting mission demands is then PD(t) = P(j) (given earlier) where j = B(t). That is, the probability that mission demands are met is simply the probability that the number of nonoperational aircraft is less than or equal to B(t).

6. The expected number of mission demands met, given k non-mission-capable aircraft is

$$E[S(t)/k] = \begin{cases} D(t) & \text{if } k \leq B(t) \\ r(t)(NA(t) - k) & \text{if } k > B(t) \end{cases} \quad (22)$$

Unconditioning by multiplying by the probability of k non-mission-capable aircraft (described in the previous paragraphs), we obtain

$$ES(t) = D(t)P(B(t)) + \sum_{k=B(t)+1}^{NA(t)} r(t)(NA(t) - k)PN_k(t) \,, \quad (23)$$

where P(B(t)) = P(j), j = B(t), and $PN_k(t)$ were described in paragraph 2 above.

7. The probability distribution of the number of mission demands met.

This is given by

$$PS_k(t) = \begin{cases} P(B(t)) & \text{if } k = D(t) \\ PN_j(t) & \text{if } k = r(t)(NA(t) - j); \\ & \quad j = B(t) + 1, \ldots, NA(t) \\ 0 & \text{otherwise} \,, \end{cases} \quad (24)$$

and allows the variance in mission demands to be determined from

$$VS(t) = D^2(t)P(B(t)) + \sum_{k=B(t)+1}^{NA(t)} (r(t)(NA(t) - k))^2 PN_k(t) - ES^2(t) \; . \tag{25}$$

Note that the mission-related measures bring into play, in a deterministic sense, additional time-varying parameters such as the number of aircraft, NA(t), the demands on all systems, D(t), and the portion of demand a single aircraft can satisfy, r(t). In the deployment of an aircraft squadron, NA(t) might represent a time-phased deployment of the aircraft as well as attrition due to losses in a conflict. The demand D(t) would be the time-dependent demand for sorties as an engagement proceeds, and r(t) would be time-varying because of a short-term capability to provide surges of flying activity.

V. AN INDENTURE MODEL FOR TIME-DEPENDENT PIPELINES

Aircraft components are classified in terms of major assemblies, which fit directly on the aircraft, and subassemblies, which fit on other assemblies. In the Air Force the major assemblies are called Line Replaceable Units or LRUs, and the subassemblies are called Shop Replaceable Units (SRUs). Further breakdowns of the assemblies are usually possible; in some cases this breakdown, or indenturing, goes through five or more levels. Assemblies and their subassemblies ("Assembly" and "component" will be used interchangeably in this section) affect the parent aircraft in different ways. A shortage of an assembly will cause a hole in an aircraft. A shortage of a subassembly will cause a hole in an assembly, which may or may not cause a shortage of the assembly and a consequent aircraft hole. Thus, in terms of effect on system performance and in terms of spare parts, subassemblies and their parent assemblies should be treated differently.

THE CLASSICAL STEADY-STATE METHOD

The classical method of treating subassemblies (see, e.g., Muckstadt, 1973) is based on a number of steady-state assumptions that do not hold true for the time-dependent case. To explain the dynamic model it will be useful to first derive the steady-state model and illustrate these assumptions. Let d_j^i be the demands per unit of time for subcomponent j when component i is repaired. Let EB_j be the expected backorders for component j and let d_i be the demands for repair of component i per unit of time. The average repair time for component i is assumed to be T_i. The first step in the classical approach is to determine the average waiting time for subcomponent j, given that it is needed in the repair of i. This is given by classical queueing theory as

$$W_j = \frac{EB_j}{d_j^i} \quad . \tag{1}$$

EB_j represents the average shortages of subassembly j that must be resupplied before new demands for j can be satisfied. In steady state, the rate of demand fulfillment is equal to the rate of demand, so that d_j^i represents the average rate at which backorders for j are satisfied. Then, EB_j/d_j^i gives the average time until the shortages are cleared, or until any new demand can be satisfied.

In the most commonly used approach, the conditional probability that the assembly i requires subassembly j when it fails is approximated by the relative demand rates for j and i,

$$P_{ij} = \frac{d_j^i}{d_i} \quad . \tag{2}$$

Under the assumption that subassembly failures are independent, the expected time that assembly i will wait for all of its subassemblies is

$$W_i = \sum_{\text{subassemblies } j} P_{ij} \cdot \frac{EB_j}{d_j^i}$$

$$= \frac{1}{d_i} \sum_{\text{subassemblies } j} EB_j \quad . \tag{3}$$

This time is then added to the assembly repair time and multiplied by the demand rate to give the average number of component i in repair plus awaiting parts (AWP). That is,

$$\lambda_i \text{ (in steady state)} = d_i \cdot (T_i + W_i)$$

$$= \underbrace{d_i T_i}_{\substack{\text{Steady-state} \\ \text{pipeline} \\ \text{without AWP}}} + \underbrace{\sum_{\text{subassemblies } j} EB_j}_{\substack{\text{Sum of} \\ \text{subassembly} \\ \text{backorders}}} \tag{4}$$

This result requires the assumption that the subcomponent failure sets are nonintersecting (which really means that no more than one subcomponent can fail or be demanded in the repair of each assembly). It also requires the assumption that there is no cannibalization of subassemblies (which would minimize the number of components awaiting parts). Finally, it assumes that a subcomponent does not appear on more than one type of parent component.

THE MODEL FOR TIME-DEPENDENT PIPELINES

The approach used for dynamic systems is similar but relaxes some of the previous assumptions. Let $A_i(t)$ represent the number of components of type i awaiting parts at time t, and let $\bar{A}_i(t)$ represent the expected value of $A_i(t)$. The mean number of components in repair and awaiting parts is then[1]

$$\hat{\lambda}_i(t) = \lambda_i(t) + \bar{A}_i(t) \tag{5}$$

[1] Let $F_A^i(t)$ represent the probability distribution of $A_i(t)$ and $F_i(t)$ be the cumulative Poisson or negative binomial distribution

As we have shown, in steady state $\bar{A}_i(t) = \bar{A}$; it is often approximated by

$$\bar{A}_i = \sum_{\substack{\text{all} \\ \text{subassemblies}}} EB_j \ . \tag{6}$$

In the Dyna-METRIC model, the determination of $\bar{A}_i(t)$ depends on whether the subassemblies are cannibalized or not.

Cannibalization with Joint Failures Allowed

When subassemblies are cannibalized, $EB_i(t)$ is determined in a manner similar to the treatment of aircraft cannibalization. Let $P^j(k)$ be the probability that shortages of the jth subassembly are less than or equal to k. Then

$$P^j(k) = \sum_{\ell=0}^{S_j(t)+k} \underbrace{\text{Probability of } \ell \text{ failures of } j}_{\substack{\text{from the Poisson or negative} \\ \text{binomial distributions}}} \tag{7}$$

---

described earlier for these components in repair. When $F_A^i(t)$ and $F_i(t)$ are independent and both Poisson, or when they are both negative binomial with the same variance-to-mean ratio, the distribution of the components awaiting parts and in repair will be of the same form with the mean $\lambda_i(t)$. Otherwise, when the distributions have different forms, convolutions between $F_i(t)$ and $F_A^i(t)$ must be performed. Although it is likely that $F_A^i(t)$ differs from $F_A^i(t)$ in distributional form, our experience has shown that the approximation of $F_A^i(t)$ by a Poisson or negative binomial distribution does not overly distort the results. We also note that convolution to obtain the resulting distribution is usually avoided in the steady-state calculations as well.

Let $P_i(k)$ be the probability that the number of nonoperational assemblies of type i (due to subassembly shortages) is less than or equal to k. Then, since this probability is the same as the probability that all subassemblies have shortages less than or equal to k,

$$P_i(k) = \prod_{\substack{\text{subassemblies} \\ j \in i}} P^j(k) \tag{8}$$

The expected number of assemblies of type i that are unavailable due to subassembly shortages is then found by summing the tail of this distribution:

$$\bar{A}_i(t) = \sum_{k=0}^{NA(t) \cdot Q_i + S_i(t)} (1 - P_i(k)) \ , \tag{9}$$

where $Q_i \equiv$ quantity of assembly i on an aircraft,

$NA(t) \equiv$ number of aircraft at time t, and

$S_i(t) \equiv$ supply level for component i at time t.

The distribution of parts of type i awaiting subassemblies is

$$F_A^i(t,k) = P_i(k) - P_i(k - 1) \ . \tag{10}$$

No Cannibalization with Joint Failures Allowed

An alternative form of $EB_i(t)$ is used when there is no cannibalization of subcomponents. The equations are similar to those used for determining the number of aircraft that are non-mission-capable under a policy of no cannibalization. In this case,

$$\bar{A}_i(t) = (NA(t) \cdot Q_i + S_i(t)) \cdot \left[1 - \prod_{\text{subassemblies}} \left(1 - \frac{EB_j \cdot (t)}{NA \cdot Q_i + S_i(t)}\right)\right]. \quad (11)$$

When there is more than one subassembly of type j on assembly i, this expected value takes on the more complicated form shown in Sec. IV. The probability distribution of subassembly shortages in the no-cannibalization case is not known although its moments can be obtained. It is expedient, although not precisely correct, to use the geometric distribution with $\bar{A}_i(t)$ as the mean.

Some Cautions and Comments on the Use of This Model

In the equations used for determining $\lambda_j(t)$, one must be careful to avoid overdrawing the influence of shortages of those components. For example, when no repair capability exists for the parent assembly, there can be no awaiting-parts quantity for that parent, even though there may have been subassembly failures. In this case the demand for the subassemblies should be deferred to the time at which repair occurs. Thus, $d_j(t)$ represents the discovered failures instead of true failures of the subcomponents.

Independence between subcomponent and parent failure distributions has been a strong assumption in much of this derivation (as it is in the steady-state case), even though most parent component failures are due to problems with the subcomponents. Clearly, the number of discovered subassembly failures cannot exceed the number of failures that exist on assemblies that have entered repair. This requires conditioning the distribution of the subassembly demands on the number of parent components in the pipeline and determining the pipeline and AWP quantities simultaneously. This means that the foregoing equations, which assumed independence, will sometimes overstate the number of assemblies in the AWP state with no cannibalization because the failed subassemblies are allowed to be distributed over some non-failed assemblies. It also means that the effects of subassembly cannibalization are sometimes overstated, since the cannibalization is allowed to take place across a wider range of assemblies than those that have failed. In general, however, the independence assumption leads to reasonable approximations since the rate of each subassembly failure is usually considerably smaller than the assembly failure rate (due to each assembly usually having large numbers of subassemblies).

Shared Subassemblies

For simplicity, the development in this section was based on an assumption that subassemblies do not fit on more than one type of parent. This is not a necessary assumption and, when violated, it can be dealt with by allocating the subassembly shortage distribution back to the potential parents based on the quantity of each type of parent in the aircraft, Q_i, the quantity of the subassembly in the aircraft, q_j, and the quantity of the subassembly on each parent, P_{ij}. Let $FA_j^T(t,k)$ be the total shortage distribution of subassembly j for the aircraft.

FA_j^T (t,k) = Probability that there are exactly k shortages of component j at time t .

We assume here that the failure rate of j is not affected by which assembly it is on, that shortages are allocated to assemblies based on relative demand, that assembly i has P_{ij} of j-type subcomponents, that there are Q_i of i-type components on the aircraft, and that there are Q_j of j-type components on the aircraft. Then the probability that there will be ℓ shortages on assembly i, when there are k ($k \geq \ell$) total shortages of subassembly j, is given by the hypergeometric distribution,

$$P(\ell,k) = \frac{\binom{Q_i \cdot P_{ij}}{\ell}\binom{Q_j - Q_i \cdot P_{ij}}{k - \ell}}{\binom{Q_j}{k}} .$$

When the total shortages k are less than ℓ, there is of course 0 probability of 1 shortages on assembly i. To illustrate this, assume $Q_i = \ell$, $P_{ij} = 2$, and $Q_j = 6$. Then 2/6 of the subcomponents j lie on assembly i. Given 4 shortages of j, the probabilities of 1 and 2 shortages on i are

$$P(1,4) = \frac{\binom{2}{1}\binom{6-2}{4-1}}{\binom{6}{4}} = \frac{8}{15}$$

$$P(2,4) = \frac{\binom{2}{2}\binom{6-2}{4-2}}{\binom{6}{4}} = \frac{6}{15} .$$

The probability of ℓ shortages of j associated with i, $FA_j^i(t,\ell)$, is removed from its dependence on the number of shortages of j by summing across the possible values of k. That is,

$$FA_j^i\,(t,\ell) = \sum_{k=\ell}^{Q_i} \frac{\binom{Q_i \cdot P_{ij}}{\ell} \cdot \binom{Q_j - Q_i \cdot P_{ij}}{k-\ell}}{\binom{Q_j}{k}} \cdot FA_j^T(t,k)$$

In case only the expected value associated with $FA_j^i(t,\ell)$ is needed (as is the case in the no-cannibalization assumption), it is easy to show that

$$\bar{A}_j^i(t) = E(FA_j^i(t,\ell) = \frac{Q_i \quad P_{ij}}{Q_j} \cdot EB_j(t)$$

Thus, the total expected shortages of j are allocated to assembly i based on the number of subassemblies on i relative to the total of j on the aircraft. The subcomponent shortage allocation described in the above paragraph is performed before computing the effect of subcomponent shortages on the parents under the cannibalization and no-cannibalization assumptions. Therefore, there is an implicit assumption of no cannibalization of subcomponents across unlike parents.

VI. A TIME-DEPENDENT PIPELINE MODEL FOR MULTIPLE ECHELONS OF REPAIR

Component repair is performed at several levels within the Air Force. Flight-line personnel remove and replace major assemblies. Squadrons or bases repair these in the field if possible or ship them to larger, centralized facilities serving several units. This section describes a mathematical model for these various echelons of repair. Except for time delays for transportation, the approach is very similar to that used to model indentured components. As before, the commonly used steady-state approach for multiple echelons is explained first, to form a basis for understanding the time-dependent approach.

THE STEADY STATE MODEL

The decision to ship a component to another echelon is based on policy (for example, perform no local repair on this component) or on random chance (inability to repair certain types of failures locally). In either case there is a decision to ship a component to another echelon of repair. At the time a component is shipped, another is ordered (usually from the same echelon). In a steady-state model, the average rate of flow to the next echelon from a unit is equaled by the average rate of flow back to the unit. Let this average flow for unit k be given by $\bar{d}_k$. Assume there is a transportation delay, T_k^F, for the retrograde movement (to the higher echelon) and a transportation delay, T_k^F, for forward shipments (from the higher echelon). The average repair time at the centralized facility is given by τ. The average number of components in repair or in transportation is then λ_k^R, given by

$$\lambda_k^R = \bar{d}_k(T_k^R + \tau) \tag{1}$$

and the average number of components in shipment to the unit is λ_k^F, given by

$$\lambda_k^F = \bar{d}_k \cdot T_k^F . \tag{2}$$

Now, if there is only one unit supported by this higher echelon, the steady-state model is quite simple. The average total pipeline for a given component is simply the base repair pipeline for the component, λ_k^B, plus the above pipeline averages:

$$\lambda_k = \lambda_k^B + \lambda_k^R + \lambda_k^F = \lambda_k^B + \bar{d}_k \; (T_k^R + \tau + T_k^F) . \tag{3}$$

Given a total stock level for the two-echelon system, the various performance measures such as backorders and fill rate can be easily computed as shown earlier.

Frequently, the total system stock is segregated into a supply level S^R to support the retrograde and repair pipeline of the higher echelon, and a supply level S_k to support the base repair and forward shipment time from the supporting facility. In this case a response delay, Δ_S, is computed and added to the forward transportation time. The response delay is approximated by computing the backorders in the retrograde and repair pipeline when the supply level is S^R, These are denoted $EB(S^R, \lambda_k^R)$. Under the continuing assumption that the system is in steady state, these backorders or shortages (actually, inability to respond to orders) are cleared at the rate $\bar{d}_k$. The time to clear all of them, and therefore the time delay in responding to a new order (since the backorders represent orders already placed), is given by

$$\Delta_S = \frac{EB(S^R, \lambda_k^R)}{\bar{d}_k} . \tag{4}$$

The forward pipeline is then increased by this time delay so that

$$\begin{aligned} \lambda_k^F &= \bar{d}_k \cdot (T_k^F + \Delta_S) \\ &= \bar{d}_k T_k^F + EB(S^R, \lambda_k^R) . \end{aligned} \tag{5}$$

It should be apparent that this separation of supply levels does not give the same performance as that computed when we consider one supply level for the entire system. That is,

$$\lambda_k = \lambda_k^B + \lambda_k^R + \bar{d}_k T_k^F \tag{6}$$

with supply level $S_k + S^R$, is not the same as

$$\lambda_k = \lambda_k^B + \bar{d}_k T_k^F + EB(S^R, \lambda_k^R) \tag{7}$$

with supply level S_k. In the first case, the retrograde and repair stock can be used to support shortages at the base <u>or</u> in the retrograde pipeline, while the second model does not permit the retrograde supply level to be used for base or unit support.

If only one base were supported, it would always make sense to place all stock in the base or unit supply level S_k, since performance is ultimately measured at the aircraft and not in the retrograde pipeline. When this is the case, the two models are the same, since

$$\begin{aligned} \lambda_k &= \lambda_k^B + d_k T_k^F + EB(0, \lambda_k^R) \\ &= \lambda_k^B + d_k T_k^F + \lambda_k^R \,. \end{aligned} \tag{8}$$

When multiple units or locations are supported by higher-echelon repair, it is necessary to consider the pipelines from each location jointly unless there is ownership of specific components. Generally, we assume that once components enter the higher echelon for repair they can be given back to any location. It is difficult, however, to determine performance of any particular unit without designating a supply level for it. When there is no retrograde supply level, the unit performance can be determined without considering the other units by adding the retrograde and repair pipelines of each individual unit and computing measures using the unit supply levels. The same performance for the system as a whole can usually be achieved at lower supply cost by providing a single retrograde plus central repair supply level for all retrograde pipelines. In this case we compute a delay, as shown earlier, using

$$\lambda^R = \sum_{k \text{ bases}} \lambda_k^R = \sum_{k \text{ bases}} \bar{d}_k(T_k^R + \tau) \tag{9}$$

and then computing

$$\Delta_S = \frac{EB(S^R, \lambda^R)}{\sum_{k \text{ bases}} \bar{d}_k}, \tag{10}$$

which is the delay in responding to any unit's order. The unit performance is then based on

$$\begin{aligned} \lambda_k &= \lambda_k^B + \bar{d}_k(T_k^F + \Delta_S) \\ &= \lambda_k^B + \bar{d}_k T_k^F + EB(S^R, \lambda^R) \cdot \frac{\bar{d}_k}{\sum_{k \text{ bases}} \bar{d}_k}. \end{aligned} \tag{11}$$

An alternative explanation of this model is that the retrograde repair shortages, $EB(S^R, \lambda^R)$, are allocated on the basis of the relative average demands placed by the units.

TIME-DEPENDENT MODEL

The dynamic multiechelon model requires introduction of the following variables:

$\lambda_k^R(t)$	Average number of components from base k in transportation to or in repair at the next higher echelon at time t.
$\lambda_k^B(t)$	Average number of components in base repair at time t.
$\lambda_k^F(t)$	Average number of components in transportation to base k at time t.
$S^R(t)$	Supply level for the retrograde pipelines at time t.
$S_k(t)$	Supply level at base k at time t.

The average shortage at time t in the retrograde transportation pipeline is given by the expected backorders,

$$EB^R(S^R(t),\lambda_k^R(t)) \; .$$

Now, if location k is the only one supported, then the shortages seen at that location are given by the excess of local repairs and unfulfilled orders (placed with the higher echelon) over the local supply level, $S_k(t)$. The average number of unfulfilled orders at time t is given by

$$\lambda_k^u(t) = \lambda_k^F(t) + EB^R(S^R(t), \lambda_k^R(t)) \ . \tag{13}$$

Depending on the repair function at the higher echelon, the average number of components in the forward pipelines can be difficult to determine, because the number must be conditional on the completion of repair and retrograde transportation. A useful alternative representation of the unfulfilled orders is given by the demands placed during one forward transportation time (since all of these must be unfulfilled) plus the shortages that existed at time $t - T_k^F$ in the retrograde plus repair portion of the system. The average unfulfilled orders are then

$$\lambda_k^u(t) = \int_{t-T_k^F}^{t} d_k^R(s)ds + EB^R(S^R(t - T_k^F), \lambda_k^R(t - T_k^F)) \tag{14}$$

and the average pipeline at location k is

$$\begin{aligned} \lambda_k(t) &= \int_{t-T_k^F}^{t} d_k^R(s)ds + EB^R(S^R(t - T_k), \lambda_k^R(t - T_k^F)) \\ &\quad + \lambda_k^B(t) \\ &= \lambda_k^u(t) + \lambda_k^B(t) \ . \end{aligned} \tag{15}$$

The development for multiple locations or squadrons served by one higher echelon repair facility is similar, except we assume that the supply level, $S_R(t)$, applies to all retrograde pipelines and that the shortages must be allocated back to the units. Thus,

$$\lambda^R(t) = \sum_{k \text{ bases}} \lambda_k^R(t) \tag{16}$$

and the average shortages in the retrograde portion of the system are

$$EB^R(S^R(t),\lambda^R(t)) \ . \tag{17}$$

These shortages should be allocated to the various locations or squadrons based on a criterion representative of shortage allocations in real situations. One criterion suggested by steady-state methods is the relative number of demands placed at time t. That is, the shortages allocated to location k at time t are given by

$$EB_k^R(t) = \frac{d_k^R(t)}{\sum\limits_{k \text{ bases}} d_k^R(t)} \cdot EB^R(S^R(t),\ \lambda^R(t)) \ . \tag{18}$$

This represents a rather volatile reallocation of shortages when the demands at the various locations go through sudden relative changes. An alternative allocation is based on the time-averaged demand at the location. Let $D_k^R(t,\tau)$ be the average demands in the interval $(t - \tau,\ t)$. That is,

$$D_k^R(t,\tau) = \int_{t-\tau}^{t} d_k^R(t)dt \quad . \tag{19}$$

Then, we can use the allocation

$$EB_k^R(t) = \frac{D_k^R(t,\tau)}{\sum\limits_{k\ \text{locations}} D_k^R(t,\tau)} \cdot EB^R(S^R(t),\ \lambda^R(t)) \quad . \tag{20}$$

The larger the value of τ, the more the shortage allocation will be influenced by (or averaged across) past orders. The appropriate value of τ is still open to empirical testing, but it seems reasonable to use a value of τ near the average time it would take the central facility to reallocate shortages. If the only means of reallocating shortages by the central facility is to not fulfill orders selectively, then the rate at which they can be allocated to any location is limited by the rate of orders placed by that location. Note that when the reallocation process is limited in this way, the shortages allocated to any location are limited by the number of orders placed by that location. (That is, the unfulfilled orders are limited by the orders placed.) The allocation of backorders using the above equation and $\tau = t$ will keep the average number of backorders allocated below the average total demands from 0 to t, since

$$BO_k^R(t) = \frac{D_k^R(t,0)}{\sum\limits_{k \text{ locations}} D_k^R(t,0)} BO^R(S^R(t),\ \lambda^R(t))$$

$$\leq \frac{D_k^R(t,0)}{\sum\limits_{k \text{ locations}} D_k^R(t,0)} \sum_{k \text{ locations}} D_k^R(t,0) = D_k^R(t,0)\ . \tag{21}$$

In our use of the model, we have approximated the distribution of allocated shortages with the Poisson or negative binomial distributions with mean value $BO_k^R(t)$. This approach is consistent with the steady-state approaches commonly in use. (See, for example, Sherbrooke, 1968(a).)

The above equations describe with sufficient generality the multiple echelon time-dependent model. Interruption of transportation and subassembly interaction with higher echelons are additional aspects of this model.

INTERRUPTION OF TRANSPORTATION

This section describes a model that has fixed or deterministic transportation times. Nondeterministic transportation times are possible within this model, but the description of unfulfilled orders is considerably more difficult. One variation that is easily accommodated is to allow interruptions of transportation for fixed periods of time. For interruption of the retrograde transportation, the pipeline calculation of $\lambda_k^R(t)$ is modified accordingly. (See the earlier section regarding pipeline calculations.) The cutoff of forward transportation affects the equation for $\lambda_k^u(t)$. If the cutoff ends before time $t - T_k^F$, or starts after t, there is no modification. If it starts before $t - T_k^F$ (at S) and ends after t, we use

$$\lambda_k^u(t) = \int_S^t d_k^R(s)ds + BO^R(S^R(S), \lambda_k^R(S)) \ . \qquad (22)$$

If it starts prior to $t - T_k^F$ and ends in the interval $[t - T_k^F, t]$, we use this also. If the cutoff starts in the interval $[t - T_k^F, t]$ and ends after t, then the original no cutoff equation applies.

INDENTURED COMPONENTS AND MULTIPLE ECHELONS

The previous section discussed the approach for indentured components. In that approach it was assumed that the demands for subassemblies occurred at the same time that demands for repairs of the parent assemblies occurred. This is a reasonable approximation for local repair, but is not correct for the repair of subassemblies at a facility remote from the aircraft. In this case the demand for repair of the major assembly occurs immediately, but the demand for a subcomponent is not known until the parent component is received and tested at the remote facility. The approach taken in this case (for higher echelons) is to delay the demand for subcomponents by the transportation time (or cutoff time plus transportation time if the retrograde transportation is cut off). Thus, the average demand from location k for a subcomponent at the remote facility is given by

$$\bar{d}_k^R(t) = \begin{cases} d_k^R(t - T_k^R) & \text{if no cutoff or if cutoff ends before } t - t_k^R \text{ or starts after } t \\ 0 & \text{if } t \text{ falls in a cut-off period or cutoff ends less than } T_k^R \text{ days prior to } t \\ \int_\alpha^\beta d_k^R(s)ds & \text{if } t - T_k^R = \beta \text{ where } \alpha \text{ and } \beta \text{ indicate the start and end of retrograde cutoff, respectively.} \end{cases} \qquad (23)$$

With this definition of demand, the indenture model of the previous section operates as described earlier.

VII. TIME-DEPENDENT OPTIMAL DETERMINATION OF SPARE PARTS TO MEET AN OPERATIONAL OBJECTIVE

Although built primarily as a readiness assessment tool, the Dyna-METRIC model, because of its analytic framework, permits the determination of spare parts required to satisfy a given level of aircraft availability. In fact, this enhances the readiness assessment by giving an answer to the question of how many additional spare parts are needed to provide a given level of readiness when goals are not achieved. (Of course, spare-parts provisioning is only one of several ways to achieve a given level of capability.) The determination of supply levels within Dyna-METRIC is separated into four phases, dealing with: spare parts to overcome queueing in test facilities, spare parts for higher echelons, spare parts for subassemblies, and spare parts of major assemblies at the squadron. Each of the latter three phases will be discussed separately in this section. The spare-parts determination for queueing is described in Sec. VIII. First, however, the general approach for a time-dependent stockage policy will be discussed.

The fact that pipelines have time-dependent probability distributions means that the optimal mix of spare components at one point in time may not be the optimal mix at another. For example, when there is no repair capability for some period of time for a subset of the components, the correct mix of spares will require substantial numbers of these components; fewer will be required if repair capability is provided at some later point in time. Clearly, there is a mix of components that will provide the desired capability at each point in time at the lowest cost. This model does not determine this cross-time optimal value for several reasons. First, the time horizon is frequently a policy variable, and the cost of the spare parts is desired for marginal increases in the horizon. That is, the decisionmakers generally want to know the incremental costs and spare parts to achieve a given capability for a slightly lengthened time horizon. Complete changes of the spare-parts mix as the time horizon increases may not be desirable, particularly when supply levels already exist. Practically, furthermore, the methodology for cross-time optimization has not been developed. The approach taken is to compute, for each time of interest, the marginal increase in spare parts to achieve a given capability _over those already input or determined for a previous time_. Thus, embedded in each subsequent time is a supply level that will achieve the desired capability at all previous times.

SUPPLY LEVELS FOR SQUADRONS OR OPERATING UNITS

Aircraft mission capability at the squadron is the primary interest and is the constraint in determining the supply level. This is expressed as a probability of not exceeding a given number of non-mission-capable aircraft at each time point of interest. That is, the decisionmaker selects the confidence of achieving a given mission-capable rate for the squadrons. In determining the supply level, the model attempts to provide enough spare parts to give the desired confidence at lowest cost at each point in time of interest. Thus, the objective function is the total cost of spare parts at the squadron. Let S_i represent the spare-parts level for component i, c_i the unit cost of component i (per-unit costs are assumed to be independent of quantity), α the desired confidence level, K_N the non-mission-capable rate not to be exceeded, and $P(K_N,S)$ the probability that the non-mission-capable rate is less than K_N given a stock level S (the vector of S_i's). Then the problem solved for each squadron or operating location at each time of interest is

$$\text{minimize} \quad \sum_{i \text{ components}} c_i S_i$$

$$\begin{aligned} \text{subject to} \quad & P(K_N,S) \geq \alpha \qquad (1)\\ & S_i \geq S_{io} \qquad \text{for each } i,\\ & S_i \text{ integer} \end{aligned}$$

where S_{io} is the input stock level or previous time optimization stock level for component i. The probability function $P(K_N,S)$ is the form given in Sec. IV for full cannibalization. That is,

$$P(K_N,S) = \prod_{i \text{ components}} p^i(Q_i \cdot K_N) \qquad (2)$$

where

$$p^i(Q_i \cdot K_N) = \sum_{k=0}^{S_i+Q_i \cdot K_N} P_i(k) \qquad (3)$$

and $P_i(k)$ is the probability of exactly k failures of component i.

The reader may observe that a necessary condition for the performance constraint to be met is to have

$$P^i(Q_i \cdot K_N) \geq \alpha \tag{4}$$

for each i (since otherwise the product of these functions, which have values less than or equal to 1, would not be greater than α).

Thus, the first step in the requirements process is to find a stock level for each component that satisfies this necessary condition independently. Once this step has been completed, the programming constraint will not usually be satisfied, and this is the point at which optimization comes into play. *Marginal analysis* is used to determine the "best" mix of additional components to achieve the desired goal. This process proceeds by investing in one additional component at a time which is selected by finding the component that gives the largest increase in the logarithm of the confidence level at the lowest cost.[1] That is, we determine

$$\frac{\Delta_i \ell n\ P(K_N, S)}{c_i}, \tag{5}$$

---

[1] The logarithm of $P(K_N,S)$ allows the constraint to be converted from a single product to a sum of terms. This separation of terms then allows marginal analysis to find an optimal or nearly optimal solution. See App. A.

where

$$\Delta_i \ \ell n \ P(K_N,S) = \ell n \ [P(K_N,S^i)/P(K_N,S)] \qquad (6)$$

and

$$S^i = (S_1,S_2\ldots,S_i + 1, \ S_{i+1}, \ \ldots) \ . \qquad (7)$$

The component for which supply is increased one unit is the one whose index solves

$$\max_i \quad \frac{\Delta_i \ \ell n \ P(K_N,S)}{c_i} \ . \qquad (8)$$

This process continues until the given confidence level is achieved. At this point the resulting supply vector S* will be an "undominated" or "efficient" solution of the base stockage problem (see Fox, 1966).[2] For most purposes an undominated solution can be considered to be optimal or nearly optimal.

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[2] To reach the undominated solution, the values of α and KN must be set to "reasonable" levels. See App. A for more discussion of this.

SUPPLY LEVEL FOR INDENTURED COMPONENTS

Supply levels for subcomponents are determined by a heuristic that attempts to weight the subcomponent need based on the relative value of shortages of parent assemblies. At the same time, it attempts to achieve a minimum-cost solution among the subcomponents and parent assemblies. Muckstadt (1980), in the mod-METRIC model, used a search of the budget space to determine an optimal solution for the case in which system performance in terms of supply backorders was to be optimized. Kotkin (1978) described efficient cost heuristics for the dual problem, which minimized cost subject to a backorder constraint. This approach uses similar heuristics for determining quantities of subassemblies that minimize cost subject to the operationally ready rate measure described in the previous subsection.

Recall that the problem of interest is to minimize

$$\sum_i c_i \; S_i$$

subject to:

$$\prod_i P^i(Q_i K_N, S_i) \geq \alpha$$

$$\text{(9)}$$

$$S_i \geq S_{io} \text{ for all } i \; .$$

The indenture problem adds the variable S_j^s, the supply level of the subassembly j. In this case the problem becomes:

$$\text{minimize} \qquad \sum_i c_i S_i + \sum_j c_j^s \, S_j^s$$

$$\text{subject to} \qquad \prod_i P^i(Q_i \cdot K_N, S_i, S^{si}) \geq \alpha \tag{10}$$

$$S_i \geq 0 \text{ for all } i$$

$$S_j^s \geq 0 \text{ for all } j \,,$$

where S^{si} represents the supply of all subcomponents that have component i as a parent. Note that the variable name $P^i(Q_i K_N, S_i)$ has been expanded to show the dependence on S^{si} by writing it as

$$P^i(Q_i K_N, S_i, S^{si}) \;.$$

With the logarithmic transformation this becomes

$$\text{minimize} \qquad \sum_i c_i S_i + \sum_i c_j^s S^s{}_j$$

$$\text{subject to} \qquad \sum_i \ell n \; P^i(Q_i K_N, S_i, S^{si}) \geq \ell n \; \alpha \tag{11}$$

$$S_i \geq 0 \quad \text{for all } i$$

$$S_j^s \geq 0 \quad \text{for all } j \;.$$

An appropriate Lagrange multiplier, γ, allows the constraint to be moved to the objective.[3] This converts problem (11) to the following form:

$$\text{minimize} \quad \sum_i c_i S_i + \sum_j c_j^s S_j^s - \gamma \left(\sum_i \ln P^i(Q_i K_N, S_i, S^{si}) - \ln \alpha \right)$$

$$\text{subject to} \quad S_i \geq 0 \quad \text{for all } i \tag{12}$$

$$S_j^s > 0 \quad \text{for all } j \,.$$

With an appropriate value of γ (we will show how to find an approximate γ shortly), this problem is equivalent to (11). When γ and S_i are known, it is relatively easy to determine S_j^s for each j. The overall heuristic is to:

1. Start with a given value of S_j^s, say S_j^{so}.
2. Determine S_i under the procedure given earlier in this section.
3. Determine an appropriate value of γ.
4. Determine S_j^s given γ.
5. Redetermine S_i given the new values of S_j^s.

---

[3] Fox (1978) and Fox and Landi (1969) discuss the existence of this multiplier. It can be thought of as a cost or penalty for not satisfying the performance constraint.

Once step 2 has been completed, the value of γ can be approximately determined by assuming that S_j^s is constant and applying an optimality condition to problem (12).[4] That is, if $S = (S_1, S_2, \ldots, S_k)$ solves (12), then

$$c_i(S_i + 1) - c_i(S_i) - \gamma \left[\ell n\, P^i(Q_i K_N, S_i + 1, S^{si}) - \ell n\, P^i(Q_i K_N, S_i, S^{si})\right] \geq 0 \quad \text{for all } i \tag{13}$$

and

$$c_i(S_i - 1) - c_i(S_i) - \gamma \left[\ell n\, P^i(Q_i K_N, S_i - 1, S^{si}) - \ell n\, P^i(Q_i K_N, S_i, S^{si})\right] \geq 0 \quad \text{for all } i \tag{14}$$

Thus,

$$\gamma \leq \frac{c_i}{\ell n\, \left[P^i(Q_i K_N, S_i + 1, S^{si}) / P^i(Q_i K_N, S_i, S^{si})\right]} \tag{15}$$

---

[4] This condition merely states that the optimal value of the vector S must give a smaller value of the objective than does any other value.

and

$$\gamma \geq \frac{c_i}{\ell n \; [P^i(Q_i K_N, S_i, S^{si})/P^i(Q_i K_N, S_i - 1, S^{si})]} \; , \tag{16}$$

which provide bounds on the value of γ. The algorithm uses the upper bound, which will err on the side of too many subassemblies (which typically cost much less than the parent assemblies). That is, it uses

$$\gamma = \min_i \frac{c_i}{\ell n \; [P^i(Q_i K_N, S_i + 1, S^{si})/P^i(Q_i K_N, S_i, \; S^{si})]} \tag{17}$$

This is particularly convenient since it is the reciprocal of the ratio used in the solving (1) at step 2, and the index solving this corresponds to the last component of supply increased in that process. (Thus, the approximate γ is immediately known because it is the marginal improvement ratio for the last stock increase.)

Given this value of γ the subassembly problem is next solved in step 4 of the heuristic. That problem is, given primary assembly stock,

$$\text{minimize} \qquad \sum_j c_j S_j^s - \gamma \left(\sum \ell n \; P^i(Q_i K_N, S_i, S^{si}) - \ell n \; \alpha \right) \tag{18}$$

$$S_j^s \geq 0 \qquad \text{for all } j \; .$$

Appendix B shows that this problem is solved approximately by finding a level of stock for each subcomponent j that satisfies

$$RR_j\ (S_j^s) \geq 1 + \frac{c_j}{\gamma\beta_j}\ , \qquad (19)$$

where $RR_j(S_j^S)$ is the ready rate for subcomponent j defined in Sec. III and B_j is a constant defined in App. B.

SUPPLY LEVELS FOR HIGHER ECHELONS

The description of multiple echelons in Sec. VI indicated that, for one base or squadron served by one higher echelon, it was more efficient to determine one system supply level rather than provide supply levels for the squadron and higher echelon separately. This subsection addresses the problem of providing spares to a higher echelon facility serving several squadrons at different locations. In this case it is usually efficient to provide a system supply level for the retrograde plus repair pipeline of the higher-echelon facility, and separate local supply levels for each of the squadrons to cover the order and shipping time of ordered components and for those repairs done at the squadron. For the centralized supply, we employ a heuristic similar to that used for indentured components:

1 Start with a given level of supply at the higher echelon, say S_i^{co}.
2. Determine S_i at the squadrons as discussed earlier (after computing the effect of central system shortages using S_i^{co}).

3. Determine an appropriate <u>set</u> of Lagrange multipliers given the solution in step 2.
4. Determine the supply level, S_i^c, at the higher echelon given the multipliers and the S_i from step 2.
5. Redetermine S_i at each squadron given the new value of central system supply.

The overall problem to be solved is[5]

$$\text{minimize} \quad \sum_i c_i S_i + \sum_i c_i S_i^c$$

$$\text{subject to} \quad \prod P_i^k(Q_i K_N, S_i^k, S_i^c) \geq \alpha \qquad \text{for all k locations served [6]} \tag{20}$$

$$S_i^k, S_i^c \geq 0 \qquad \text{for all k and i.}$$

Using Lagrange multipliers and the logarithmic transformations, this problem becomes

$$\text{minimize} \quad \sum_{k \text{ locations}} \sum_i c_i S_i^k + \sum_i c_i S_i^c$$

$$- \sum_{k \text{ locations}} \gamma_k \left(\sum_i \ell n \; P_i^k(Q_i K_N, S_i^k, S_i^c) - \ell n \; \alpha \right)$$

$$S_i^k, S_i^c \geq 0 \qquad \text{for all k and i.} \tag{21}$$

---

[5] The new notation indicates the dependence of the constraint on the central system supply rather than the subcomponents, and also shows the dependence on a location dependent supply level, S_k.

[6] The probability expression in the constraint is the same as that in problem (1) and problem (2) of this section and described earlier in Sec. IV.

With the central system supplies, S_i^c, held constant (at S_i^{co}), this decomposes into subproblems involving only the component i at location k and the squadron or base supply:

$$\min \quad c_i S_i^k + \gamma_k(\ln P_i^k(Q_i K_N, S_i^k, S_i^{co}) - \ln \alpha) \tag{22}$$

$$S_i^k \geq 0 .$$

The value of γ_k is available after this problem is solved through marginal analysis, as shown in the last subsection. (In fact, it is the <u>same</u> Lagrange multiplier that is used in the indenture problem for the squadron or base.) At step 4 the problem solved is

$$\min \quad c_i S_i^c - \sum_k \ln P_i^k(Q_i K_N, S_i^k, S_i^c) \cdot \gamma_k \tag{23}$$

$$S_i^c \geq 0 .$$

With the base-level supply, S_i^k, held constant at the value given in step 2, this problem is solved by increasing the central-system spares level, S_i^c, until the objective function no longer decreases.

The reader may observe that since the model of the higher echelon, described in Sec. VI, actually has that echelon affecting the squadron's performance after a time delay equal to the forward transportation time, the point in time for determination of a supply level is ambiguous for several locations with different transportation times. The function $P_i^k(Q_i, K_N, S_i^k, S_i^c)$, for each location accommodates these separate times, and it is necessary only to assume that S_i^c remains constant during the different delay periods.

The calculation of subassembly supply levels at the higher echelon interacts with the calculation of S_i^c. Given the subproblem (23), we can break this into one involving identical components:

$$\min \quad c_i S_i^c - \sum_k \gamma_k \, \ell n \, P_i^k(Q_i K_N, S_i^k, S_i^c, S_i^{sc})$$

$$\text{subject to} \qquad S_i^c, S_i^{sc} \geq 0 \tag{24}$$

$$\text{where} \qquad S_i^{sc} = (S_1^{sc}, S_2^{sc}, \ldots, S_k^{sc}) \ ,$$

a vector of subcomponents associated with component i at the higher echelon. Actually, since the subcomponents may affect more than one major assembly, we need to solve

$$\min \quad \sum_i c_i S_i^c - \sum_i \sum_k \gamma_k \, \ell n \, P_i^k(Q_i K_N, S_i^k, S_i^c, S_i^{sc})$$

$$S_i^c, S_i^{sc} \geq 0 \qquad \text{for all } i. \tag{25}$$

This problem is made separable in terms of the subcomponents by holding S_i^{sc} and S_i^c fixed, and differentiating with respect to the subcomponent shortage, $B_j(S_j^{sc})$, as shown earlier. We then obtain, as illustrated in App. B, subproblems with the form

$$\min \quad c_j^s S_j^s - \sum_k \gamma_k \, \beta_j^k \, B_j(S_j^{sc})$$

$$S_j \geq 0 \tag{26}$$

where

$$\beta_j^k = \sum_{\substack{\text{components}\\ \text{which}\\ \text{have}\\ \text{subassembly } i}} \frac{1}{P_i^k(Q_i K_N, S_i^k, S_i^c, S_i^{sc})} \cdot \frac{\partial P_i^k(Q_i K_N, S_i^k, S_i^c, S_i^{sc})}{\partial \; B_j(S_j^{sc})} \tag{27}$$

The subproblem can be easily solved by increasing the value of S_j^{sc} until the objective function no longer decreases. The complete algorithm is:

1. Start with a given level of supply at the higher echelon, say S_i^{co} and S_i^{sco} for all components and subcomponents.
2. Determine S_k at each of the k squadrons (after computing the effect of central system shortage using S_j^{co} and S_j^{sco}).
3. Determine an appropriate <u>set</u> of Lagrange multipliers given the solution in step 2.

4. Given the value of S_k from step 2, and the initial value of S_j^{co} for all subcomponents, determine the partial derivative β_j^k with respect to each subcomponent for each location.
5. Determine the optimal value of S_j^{sc} for the central system.
6. Given this value, determine the optimal value of S_i^c for the central system using γk and subproblem 23.
7. Given S_i^c and S_j^{sc}, redetermine S_i^k.

VIII. LIMITED SERVICE CAPACITY

A very basic assumption in Hillestad and Carrillo (1980) and in the previous sections of this report is that sufficient slack service capacity exists to avoid queueing in the repair of components. Pokress (1977) has shown that the infinite-server assumption is valid as long as average demands remain less than 80 percent of service capacity. But surges of aircraft flying may cause increased failures such that the component repair resources (manpower, facilities, or test equipment) become overloaded. When this overload goes on indefinitely, the reparable component queues can be very large, but when there is slack capacity after a surge, the queue can be worked down. We are interested in the time-dependent mission capability of aircraft, that is, how fast the mission capability degrades and recovers from such variations in demand. The steady-state results of queueing theory for limited-server queues do not provide information about this transient behavior, and the few analytical solutions to limited-server queues in the transient case are too restrictive to capture the complexities of multiple-use test equipment, priority scheduling, etc.

Dyna-METRIC approximates the queueing probability distribution with a simple simulation. The basic approach has been to decompose the queueing and service portion of the pipeline. That is, the total average pipeline of a given component is assumed to be composed of

$$\bar{\lambda}(t) = \lambda(t) + B(t) \ ,$$

where $\lambda(t)$ is the average pipeline as computed in Sec. II, and $B(t)$ is the average number of components in a component repair queue. (Note that Air Force steady-state inventory calculations assume that the repair time includes both average service time and average queueing time.)

The determination of $B(t)$ is currently done by a mean-value simulation that fails components and schedules them to the limited servers, based on a rule that places highest priority for repair of those components with the greatest shortages. The simulation considers the availability of test equipment or specialists in scheduling the repairs and creates an estimate of $B(t)$ for use by the remaining processes in Dyna-METRIC. Only the average queue size, $B(t)$, is stored. The probability distributions are not estimated in the simulation; instead, they are assumed to be the same as those for $\lambda(t)$. The simulation is discussed in Pyles et al. (1981) and will not be discussed further here.

Two aspects of limited service capability are modeled analytically, however, and will be discussed further in this section. These are the availability of failing test equipment and additional spare parts required to avoid shortages when there is not sufficient service capacity.

TEST EQUIPMENT AVAILABILITY

When test equipment is unreliable, its availability is an important consideration in periods of high transient demand (such as the onset of a conflict) since both the queues and the required inventory of spare parts to achieve a desirable mission capability may be quite large. In the simulation to determine queue sizes, it is necessary to know the probability that a given test stand is available at a random point in time. The following assumptions, based on empirical studies of certain types of test equipment by Gebman, Shulman, and Moore (forthcoming) allow us to use a dynamic pipeline model similar to those in Sec. II to determine this probability. These are:

1. The rate at which test-stand part-shortages are generated at any point in time is known to be $d^T(t)$. (Note that only the overall rate of shortage is represented, and that particular components of the test stands are not distinguished in this model.)
2. The resupply for test-stand shortages is given by the probability function $\bar{F}^T(t,s)$, the probability that a shortage at time s remains at time t.
3. Shortages of test-stand components are consolidated on the smallest number of stands possible.
4. A shortage of a test-stand component has an associated probability, p_i, that an aircraft component i can still be tested despite that shortage.
5. Shortages occur on no more than one of any test-stand component type at a time, so that the number of test stands not operational after cannibalization is never more than one stand.

Using these assumptions, we can determine the average number of test-stand shortages with

$$\lambda^T(t) = \int_0^T d(s)\bar{F}^T(t,s)ds \ . \tag{1}$$

The probability of having exactly k shortages is then, under a Poisson rate of shortage generation,

$$P^T(t,k) = \frac{\lambda^T(t)^k e^{-\lambda^T(t)}}{k!} \tag{2}$$

The probability that the "hangar queen" test stand can repair component i, given k shortages, is then

$$P_i^T(t,k) = \frac{p_i^k \lambda^T(t)^k e^{-\lambda^T(t)}}{k!} \tag{3}$$

$$= e^{-(1-p_i)\lambda(t)} \quad \frac{p_i^k \lambda^T(t)^k e^{-p_i \lambda^T(t)}}{k!} .$$

The probability that the hangar queen test stand can repair component i at time t is

$$P_i^T(t) = \sum_{k=0}^{\infty} P_i^T(t,k) = e^{-(1-p_i)\lambda^T(t)} \sum_{k=0}^{\infty} p_i^k \lambda^T(t)^k e^{-p_i \lambda^T(t)} \tag{4}$$

$$= e^{-(1-p_i)\lambda^T(t)}$$

This probability is used in the simulation to determine availability of the test equipment for testing certain types of components. Tests are first scheduled on the hangar queen, utilizing what capability exists, are then scheduled on the remaining pieces.

SPARE AIRCRAFT COMPONENTS TO OVERCOME QUEUES

Additional spare-parts inventory levels are determined for cases in which there is insufficient service capacity. These spare parts allow the aircraft to remain at the desired level of mission capability even though certain components are not repaired. Clearly, this is an option only for a limited time period, since it is never possible to provide spares for insufficient service capacity over an unlimited time span. The approach taken is to determine whether the average service capacity will be exceeded over the time horizon of interest, and when it is, to "buy out" enough low-cost components to allow the average service capacity to match the average service demands. Thus, the procedure is as follows:

1. Determine the average service capacity of each server type for the period.
2. Determine the average service demands for each server type for the period.
3. Determine, for those server types with insufficient capacity, the component with the lowest ratio of cost to test-time. Buy spares for that component until either service capacity is adequate (assuming that each spare represents a test that does not have to be performed during the period) or until the average

demands for the component for the period are covered. Continue until adequate service capacity exists for each server type.

These "server inadequacy" spares are used along with input supply levels as the starting basis for supply-level determination in the previous sections.

Appendix A

CONVERGENCE OF THE MARGINAL ANALYSIS APPROACH

This appendix describes conditions of convergence for the basic supply optimization algorithm of Sec. VII, problem (1). That problem is restated here as follows:

$$\text{minimize} \qquad CS = \sum_{i \text{ components}} c_i S_i$$

$$\text{subject to} \qquad P(K_N, S) \geq \ell \qquad (1)$$

$$S_i \geq S_{io} \quad \text{for each } i$$

$$S_i \text{ integer} \; .$$

The marginal analysis algorithm attempts to find an "undominated" solution to this problem.

An undominated solution is defined by Fox (1966) as one for which

$$CS < CS^* \Rightarrow P(K_N, S) < P(K_N, S^*) \qquad (2)$$

and

$$CS \leq CS^* \Rightarrow P(K_N, S) \leq P(K_N, S^*) \qquad (3)$$

That is, since by definition $P(K_N,S^*) \geq \alpha$, any solutions that are better than S^* must have a value for $P(K_N,S)$ that lies between $P(K_N,S^*)$ and α. If $P(K_N,S^*) = \alpha$, then S^* is an optimal solution. Since the value of S^* is arrived at by increasing some component's supply level by one unit at the last step, we know that $P(K_N,S^*)$ is likely to be close to α.

Let S^* be arrived at on the kth iteration and S^{k-1} be the previous value of S. Let S be the optimal solution to problem (1). Then we also know that

$$P(K_N,S^{k-1}) < \alpha \leq P(K_N,\hat{S}) \leq P(K_N,S^*) \ . \tag{4}$$

The reason that logarithms are used is to guarantee convergence to an undominated solution. Marginal analysis will provide an optimal solution to the problem

$$\text{minimize} \quad \sum_i c_i S_i$$

subject to $\phi(S) \geq \alpha$

$$S_i \geq 0 \text{ for all } i$$

if $\phi(S)$ is separable into $\phi(S) = \sum_i \phi_i(S_i)$ and each of these is a concave, increasing function. See Fox (1966). Clearly, the function

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$$P(K_N,S) = \prod_i P^i(Q_i \cdot K_N) \tag{6}$$

is not separable. However, we note that

$$P(K_N,S) \geq \alpha \Longleftrightarrow \ln P(K_N,S) \geq \ln \alpha \tag{7}$$

and

$$P(K_N,S) < \alpha \Longleftrightarrow \ln P(K_N,S) < \ln \alpha \tag{8}$$

when $P(K_N,S)$ is an increasing monotonic function of S; therefore an undominated solution to

$$\text{minimize } \sum c_i S_i$$

subject to:

$$\ln P(K_N,S) \geq \ln \alpha \tag{9}$$

$$S \geq 0$$

is an undominated solution to (1). Furthermore,

$$\ell n\ P(K_N,S) = \ell n \prod_i P^i(Q_i K_N) = \sum_i \ell n\ P^i(Q_i K_N)\ , \tag{10}$$

so that we have a separable problem. It remains to show when $\ell n\ P^i(Q_i K_N)$ is concave. Concavity of a function $f(S_i)$ on integer values of S_i is defined by

$$f(S_i) - f(S_i - 1) \geq f(S_i + 1) - f(S_i)\ . \tag{11}$$

Let

$P^i(Q_i K_N, S_i) \equiv P^i(Q_i K_N)$ to show the dependence on S_i.

If $P^i(Q_i K_N, S_i)$ is concave, then $\ell n\ P^i(Q_i K_N, S_i)$ will also be concave.

Now,

$$P^i(Q_i K_N, S_i) = \sum_{k=0}^{Q_i K_N + S_i} P_i(k)\ . \tag{12}$$

Then

$$P^i(Q_i K_N, S_i) - P^i(Q_i K_N, S_i - 1) = P_i(Q_i K_N, S_i)\ , \tag{13}$$

and

$$P^i(Q_i K_N, S_i + 1) - P^i(Q_i K_N, S_i) = P_i(Q_i K_N, S_i + 1) \ . \qquad (14)$$

Thus, we have concavity if

$$P_i(Q_i K_N, S_i + 1) \leq P_i(Q_i K_N, S_i) \ , \qquad (15)$$

The recursive equations for the two distributions in Dyna-METRIC are:

$$P_i^{k+1} = \frac{\lambda(t)}{(k+1)} P_i^k \qquad \text{Poisson} \qquad (16)$$

$$P_i^{k+1} = \frac{r+k}{k+1} \cdot \frac{p}{q} P_i^k \qquad \text{Negative Binomial} \qquad (17)$$

where

$$r = \frac{\lambda(t)}{q-1} \ ,$$

q = variance-to-mean ratio, and

$p = q - 1$.

For each of these distributions we see that if $k > \lambda(t)$, then $P_i^{k+1} \leq P_i^k$ and the conditions for concavity are satisfied. Thus, the separable terms of the constraint

$$\ell n \ P^i(Q_i K_N, S_i)$$

are concave for $S_i+Q_iK_N > \lambda_i(t)$. For a reasonable value of confidence (that is, for a reasonable value of α) and mission non-capability (K_N), we can therefore expect the functions to be concave and the conditions for convergence to be satisfied.

Appendix B

THE INDENTURED SUBPROBLEMS

This appendix describes the derivation and form of the indentured subproblems for optimization problem (18) in Sec. VII.

Given primary assembly stock, we wish to minimize

$$\sum_j c_j S_j^s - \gamma \left(\sum_i \ell n \; P^i(Q_i K_n, S_i, S^{si}) - \ell n \; \alpha \right) \qquad (1)$$

$$S_j^s \geq 0 \qquad \text{for all } j \; ,$$

where

$$P^i(Q_i K_N, S_i, S^{si}) = \sum_{k=0}^{Q_i K_N + S_i} p^i(k, \lambda_i(t) + B(S^{si})) \; . \qquad (2)$$

The function $B(S^{si})$ represents the average shortage allocated to component i due to shortages in its subcomponents. As shown in Sec. V, the worst case is to assume that the sum of backorders of the subcomponents contributes directly to additional primary components in the pipeline. (This would not be the case if cannibalization of subcomponents were performed.) Again, we have used this conservative approach because generally the much lower cost of subassemblies allows a little more leeway in "overbuying." Under this assumption we have

$$B(S^{si}) = \sum_{j=1}^{N \text{ subassemblies}} B_j(S_j^s) \; . \qquad (3)$$

To create a separable problem for the subassemblies, we use a linear approximation of $P_i(Q_iK_N, S^i, S^{si})$ with respect to $B(S^{si})$. Thus,

$$P^i(Q_iK_N, S_i, S^{si}) \approx \sum_{j=1}^{N} \frac{\partial P^i(Q_iK_N, S_i, \bar{S}^{si})}{\partial B_j(S_j^s)} \cdot (B_j(S_j^s) - B_j(\bar{S}_j^s))$$

$$+ P^i(Q_iK_N, S_i, \bar{S}^{si}) . \tag{4}$$

Leaving out the constant terms and substituting the approximation into problem (4) gives the revised problem:

$$\min \sum_j c_j^s S_j^s - \gamma \sum_i \frac{1}{P^i(Q_iK_N, S_i, S^{si})} \sum_j \frac{\partial P^i(Q_iK_N, S_i, \bar{S}^{si})}{\partial B_j(S_j^s)} \cdot B_j(S_j^s) \tag{5}$$

$$S_j \geq 0 \text{ for all } j .$$

This separates into subproblems for each subcomponent in the form

$$\min c_j^s S_j^s - \gamma \left[\sum_i \frac{1}{P^i(Q_iK_N, S_i, S^{si})} \cdot \frac{\partial P^i(Q_iK_N, S_i, \bar{S}^{si})}{\partial B_j(S_j^s)} \right] \cdot B_j(S_j^s) \tag{6}$$

$$S_j \geq 0 \text{ for all } j .$$

Let β_j be the value of the term in brackets. Then we have the problem,

$$\min c_j^s S_j^s - \gamma \beta_j B_j(S_j^s)$$

$$S_j \geq 0, \tag{7}$$

which is easily solved (once α and β_j are known) by increasing S_j^s from 0 until the objective function no longer decreases. The result is the amount of subassembly stock and completes step 4. The value of β_j must be determined through perturbation of S_j^s in $P^i\ (Q_i K_N, S_i, S^{si})$ except for the case of the Poisson distribution. In this case it is easy to show that

$$\frac{\partial P^i(Q_i K_N, S_i, \bar{S}^{si})}{\partial B_j(S_j^s)} = -\frac{(\lambda_i(t) + B(S^{si}))^{Q_i K_N + S_i}}{(Q_i K_N + S_i)\,!}\, e^{-\left(\lambda_i(t) + B(S^{si})\right)} \ . \tag{8}$$

Furthermore, for the Poisson case, the solution to (7) can be converted to a ready-rate criterion on S_j^s. The solution to (7) requires that

$$c_j^s(S_j^s + 1) - c_j^s S_j^s - \gamma\ \beta_j[B_j(S_j^s + 1) - B_j(S_j^s)] \geq 0\ . \tag{9}$$

For the Poisson distribution, the difference in brackets is merely $[RR_j(S_j^s) - 1]$. This gives us the criterion

$$\gamma\beta_j[RR_j\ (S_j^s) - 1] \leq c_j^s\ .$$

Since β_j is negative, we get the criterion

$$RR_j(S_j^s) \geq 1 + \frac{c_j^s}{\gamma\beta_j}\ , \tag{10}$$

which implies that the solution to (7) can be obtained by increasing the stock of component j until it has a ready rate satisfying this criterion.

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